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# ILMIY AXBOROTNOMA

## НАУЧНЫЙ ВЕСТНИК
## SCIENTIFIC JOURNAL

## ANIQ VA TABIIY FANLAR SERIYASI

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Fizika, Kimyo, Biologiya, Geografiya

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# ON EIGENVALUES OF THE DISCRETE SCHRÖDINGER OPERATOR ON A LADDER OF ONE-DIMENSIONAL SQUARE LATTICE

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**Abstract.** We consider a discrete Schrödinger operator $H$ on a ladder of *one*-dimensional square lattices with a self-adjoint matrix-valued potential whose entries are $\lambda\cos(x-s)$ and $\mu\cos(x-s)$. We investigate conditions for the existence of eigenvalues and describe their exact locations in terms of the parameters $\lambda$ and $\mu$.

**Key words:** Discrete Schrödinger operator , essential spectrum, Ladder, lattice, spectrum.

## Bir o'lchamli kvadrat panjara zinasida diskret Shredinger operatorining xos qiymatlari

**Annotatsiya.** Mazkur ishda bir -o'lchamli kvadrat panjaralar zinasida aniqlangan, elementlari $\lambda\cos(x-s)$ va $\mu\cos(x-s)$. W dan iborat bo'lgan o'z-o'ziga qo'shma matritsaviy potensialga ega diskret Shredinger operatori ko'rib chiqiladi. Xos qiymatlarning mavjudlik shartlari tadqiq etiladi hamda ularning aniq joylashuvi $\lambda$ va $\mu$ parametrlar orqali tavsiflanadi.

**Kalit so'zlar:** Diskret Shredinger operatori, kvadrat panjara, muhim spektr, spektr, zina (ladder).

## О собственных значениях дискретного оператора Шрёдингера на лестнице одномерной квадратной решётки

**Аннотация.** В настоящей работе рассматривается дискретный оператор Шрёдингера на лестнице -мерных квадратных решёток с самосопряжённым матричным потенциалом, элементы которого задаются параметрами $\lambda\cos(x-s)$ и $\mu\cos(x-s)$. Исследуются условия существования собственных значений, а также описывается их точное расположение в зависимости от параметров $\lambda$ и $\mu$.

**Ключевые слова:** дискретный оператор Шрёдингера, спектр, существенный спектр, квадратная решётка, естница.

**1. Introduction.** The existence conditions of eigenvalues of the discrete Schrödinger operators with a $\delta$ potential is given in [1]. In the paper [2] the non-local discrete Schrödinger operators with a $\delta$ potential is considered. The authors are discussed the behavior of the spectral edges (embedded eigenvalues and resonances) of the operator at the two ends of the continuous spectrum. An exact expression for the number of discrete eigenvalues of a discrete Laplacian with a finitely supported potential is described in [5]. K. Ando et al. [8] described the Schrödinger operators on square, triangular, hexagonal, Kagome, diamond, subdivision lattices and the spectral properties of these Schrödinger operators with compactly supported potentials was studied. The inverse scattering for discrete Schrödinger operators with compactly supported potentials on $Z^d$ and on the hexagonal

lattice were studied in [3], [4] in partially, in these papers, the discreteness of embedded eigenvalues of these operators was proved.

The absence of embedded eigenvalues for Schrödinger operators with exponentially decaying potentials was proved in [7] using the unique continuation property. Using basic tools from several complex variables and algebraic geometry, in particular Hilbert's Nullstellensatz, Isozaki and Morioka [6] established a Rellich–Vekua type theorem for Schrödinger operators on the square lattice in the case of finitely supported potentials. The Rellich–Vekua type uniqueness theorem, the unique continuation theorem, and the non-existence of eigenvalues embedded in the continuous spectrum for discrete Schrödinger operators with compactly supported potentials on periodic lattices including the square, triangular, and hexagonal lattices were studied in [8], where Kagome lattice and the subdivison of the square lattice, there are examples of Hamiltonians with compactly supported potentials having embedded eigenvalues. In [9] is extented the results in [8] for Schrödinger operators with exponentially decreasing potentials on a class of lattices including square, triangular, hexagonal lattices and their ladders. The authors discussed the unique continuation property and provided a counterexample for Hamiltonians on a ladder of a $d$ -dimensional square lattice with a diagonal matrix potential whose entries are identical scalars, exhibiting an embedded eigenvalue.

In the present paper, we investigate spectral properties of a discrete Schrödinger operator $H$ on a ladder of *one* -dimensional square lattices with a self-adjoint matrix-valued potential whose entries are $\lambda\cos(x-s)$ and $\mu\cos(x-s)$. The main aim of this paper is to determine conditions for the existence of eigenvalues and to describe their exact locations in terms of the parameters $\lambda$ and $\mu$ for the case $d=1$. In particular, we clarify the existence of eigenvalues embedded in the essential spectrum of $H$.

***2. Basic statements and The main results.***

**Ladder of one-dimensional square lattice in** $\mathsf{R}^2$. Let $\mathsf{L}_0=\{(n_1,0): n_1\in\mathsf{Z}\}$. Letting $p_1=(0,0)$ *and* $p_2=(0,1)$, we put

$$\mathsf{V}=\mathsf{V}_1\cup\mathsf{V}_2,$$

where $\mathsf{V}_j=p_j+\mathsf{L}_0$, $j=1,2$. For $v_j\in\mathsf{V}$ the adjacent relation is defined as

$$N_{v_j}=\{\omega\in\mathsf{V}:|\ \omega-v_j\ |=1\},\qquad j=1,2.$$

Let $\lambda_2(\mathsf{V})$ be the set of complex valued functions $f$ on $\mathsf{V}$ satisfying

$$\|\ \hat f\ \|^2_{deg}:=\sum_{v\in\mathsf{V}}|\ \hat f(v)|^2\ deg(v)<\infty,\quad \deg(v)=\#N_v=3.$$

Then $\lambda_2(\mathsf{V})$ is the Hilbert space equipped with the inner product

$$(\hat f,\hat g)_{deg}=\sum_{v\in\mathsf{V}}\hat f(v)\overline{\hat g(v)}deg(v).$$

The discrete Laplacian $\hat\Delta$ on $\lambda_2(\mathsf{V})$ is defined as [8]

$$(\hat\Delta_{\Gamma_0}\hat u)(n)=\frac{1}{3}\begin{pmatrix}\hat u_2(n)+\hat u_1(n+1)+\hat u_1(n-1)\\ \hat u_1(n)+\hat u_2(n+1)+\hat u_2(n-1)\end{pmatrix},\quad n\in\mathsf{Z}.$$

We introduce a discrete Schrödinger operator $\hat H=\hat H_0-\hat V$ on $\lambda_2(\mathsf{V})$ by

$$(\hat H\hat u)(v)=(\hat H_0\hat u)(v)-\hat V(v)\hat u(u),\quad \hat H_0=-\hat\Delta_{\Gamma_0},\quad v\in\mathsf{V},$$

where $\hat H_0=-(\hat\Delta_{\Gamma_0}\hat u)(n),\quad n\in\mathsf{Z}$, $\hat V$ is a multiplication operator on $\lambda_2^{(2)}(\mathsf{V})$ is represented as the operator of multiplication by real-valued $2\times2$ matrix

$$V(n)=\begin{pmatrix}\lambda\hat V(n) & \mu\hat V(n)\\ \mu\hat V(n) & \lambda\hat V(n)\end{pmatrix},\qquad \hat V(n)=\begin{cases}\pi, & \text{for} n=\pm1,\\ 0, & \text{for} n\neq\pm1,\end{cases}$$

$\lambda$ and $\mu$ are real numbers with $(\lambda,\mu)\neq(0,0)$. Then $\hat{H}$ and $\hat{H}_0$ are bounded, self-adjoint operators on the Hilbert space $\lambda_2^{(2)}(\mathsf{V})$.

Let $L_2(\mathsf{T})$ be the Hilbert space of square-integrable functions defined on $\mathsf{T}=(-\pi,\pi]$ and $L_2^{(2)}(\mathsf{T})$ denote the Cartesian product of two copies of $L_2(\mathsf{T})$. Let

$$\mathsf{F}: L_2^{(2)}(\mathsf{T})\to\lambda_2^{(2)}(\mathsf{Z}),\quad (\mathsf{F}f)_\alpha(n)=\frac{1}{(2\pi)^{\frac{1}{2}}\sqrt{deg_j}}\int_{\mathsf{T}}e^{-inx}f_\alpha(x)dx,\alpha=1,2,$$

be the standard Fourier transformation.

The momentum representation $H=\mathsf{F}^{-1}\hat{H}\mathsf{F}$ of $\hat{H}$ acts in the Hilbert space $L_2^{(2)}(\mathsf{T})$ as

$$H=H_0-V,$$

where $H_0$ is a multiplication operator by matrix

$$H_0(x)=\frac{-1}{3}\begin{pmatrix}2\cos x & 1\\ 1 & 2\cos x\end{pmatrix}$$

and $V$ is an operator matrix

$$V=\begin{pmatrix}\lambda\mathbf{v} & \mu\mathbf{v}\\ \mu\mathbf{v} & \lambda\mathbf{v}\end{pmatrix}$$

with the integral operator

$$\mathbf{v}f=\int_{\mathsf{T}}\cos(x-s)f(s)ds,\quad f\in L_2(\mathsf{T}).$$

The spectrum $\sigma(H_0)$ of the operator $H_0$ [4] is

$$\sigma(H_0)=[-1,1].$$

Because $V$ is a compact operator, Weyl's theorem for the essential spectrum [11] yields

$$\sigma_{ess}(H)=\sigma(H_0),$$

where $\sigma_{ess}(H)$ is essential spectrum of $H$.

Let

$$\tilde{H}=\begin{pmatrix}H_1 & 0\\ 0 & H_2\end{pmatrix},$$

where

$$H_1=H_{10}-(\lambda+\mu)\mathbf{v},\quad H_2=H_{20}-(\lambda-\mu)\mathbf{v},$$

$H_{10}$ and $H_{20}$ are multiplication operators on $L_2(\mathsf{T})$ by functions

$$E_{10}(x)=-\frac{2}{3}\cos x-\frac{1}{3}\quad and\quad E_{20}(x)=-\frac{2}{3}\cos x+\frac{1}{3},$$

respectively.

**Lemma 2.1** *The operator $\tilde{H}$ is unitarily equivalent to the operator H.*

Since the operators $H$ and $\tilde{H}$ are unitarily equivalent, an investigation eigenvalue problem for $H$ reduces to study eigenvalue problems for $H_1$ and $H_2$, since

$$\sigma(H)=\sigma(H_1)\cup\sigma(H_2).$$

The Weyl theorem for the essential spectrum [11] implies that the essential spectrum $\sigma_{ess}(H_1)$ and $\sigma_{ess}(H_2)$ of the operators $H_1$ and $H_2$ coincide with the range of the functions $E_{10}$ and $E_{20}$, respectively, i.e.

$$\sigma_{ess}(H_1)=[-1,\frac{1}{3}] \quad and \quad \sigma_{ess}(H_2)=[-\frac{1}{3},1].$$

We introduce the following functions

$$\Delta_\alpha(z)=\Delta_\alpha^e(z)\Delta_\alpha^o(z), \quad z\notin\sigma_{ess}(H_\alpha), \quad \alpha=1,2,$$

$$\Delta_1^e(z)=1-(\lambda+\mu)I_1(z), \quad \Delta_1^o(z)=1-(\lambda+\mu)J_1(z),$$

$$\Delta_2^e(z)=1-(\lambda-\mu)I_2(z), \quad \Delta_2^o(z)=1-(\lambda-\mu)J_2(z),$$

where

$$I_\alpha(z)=\int_T\frac{\cos^2 xdx}{E_{\alpha 0}(x)-z}, \qquad J_\alpha(z)=\int_T\frac{\sin^2 xdx}{E_{\alpha 0}(x)-z}.$$

**Lemma 2.2** *The number* $z\in\mathbb{R}\setminus\sigma_{ess}(H_\alpha),\alpha=1,2,$ *is an eigenvalue of the operator* $H_\alpha$ *if and only if* $\Delta_\alpha(z)=0.$

This lemma allows to describe the location of the eigenvalues of the operator $H_\alpha$ with respect to the parameters $\lambda$ and $\mu$.

**Lemma 2.3** *For any* $\lambda+\mu\neq 0,$ *the operator* $H_1$ *has two eigenvalues* $z_1$ *and* $z_2$ *lying on* $\mathbb{R}\setminus[-1,\frac{1}{3}]$ *and* $H_1$ *has no any eigenvalue for* $\lambda=-\mu$. *Moreover,*

(1) If $\lambda+\mu>0$, then $z_1,z_2\in(-\infty,-1)$;

(2) If $\lambda+\mu<0,$ then $z_1,z_2\in(\frac{1}{3},\infty)$. More precisely,

(i) If $\frac{1}{\pi(6-4\sqrt{3})}<\lambda+\mu<0,$ then $z_1,z_2\in(\frac{1}{3},1).$ Moreover, in this case the eigenvalues $z_1,z_2$ of $H_1$ lie in the essential spectrum $\sigma_{ess}(H_2)$ ;

(ii) If $\lambda+\mu=\frac{1}{\pi(6-4\sqrt{3})}$, then $z_1=1$ and $z_2\in(\frac{1}{3},1)$, i.e. $z_1=1$ is the threshold of $\sigma_{ess}(H_2)$;

(iii) If $\frac{1}{\pi(3\sqrt{3}-6)}<\lambda+\mu<\frac{1}{\pi(6-4\sqrt{3})}$, then $z_1\in(1,\infty)$ and $z_2\in(\frac{1}{3},1)$ ;

(iv) If $\lambda+\mu=\frac{1}{\pi(3\sqrt{3}-6)}$,then $z_1\in(1,\infty)$ and $z_2=1$ , i.e. $z_2=1$ is the threshold of $\sigma_{ess}(H_2)$;

(v) If $\lambda+\mu<\frac{1}{\pi(3\sqrt{3}-6)}$, then $z_1,z_2\in(1,\infty).$

**Lemma 2.4** *For any* $\lambda-\mu\neq 0,$ *the operator* $H_2$ *has two eigenvalues* $z_3,z_4$ *lying on* $\mathbb{R}\setminus[-\frac{1}{3},1]$ *and* $H_2$ *has no any eigenvalue for* $\lambda=\mu$. *Moreover,*

(1) If $\lambda-\mu<0$ ,then $z_3,z_4\in(1,\infty)$;

(2) If $\lambda-\mu>0,$ then $z_3, z_4 \in (-1,\infty).$ More precisely,

(i) If $0<\lambda-\mu<\dfrac{1}{\pi(4\sqrt{3}-6)}$ ,then $z_3, z_4 \in (-1,-\dfrac{1}{3});$ . Moreover, in this case $z_3, z_4$ lie in the essential spectrum $\sigma_{ess}(H_1)$;

(ii) If $\lambda-\mu=\dfrac{1}{\pi(4\sqrt{3}-6)}$, then $z_3=-1$ and $z_4 \in (-1,-\dfrac{1}{3}),$ i.e. $z_3=-1$ is the threshold of $\sigma_{ess}(H_1)$;

(iii) If $\dfrac{1}{\pi(4\sqrt{3}-6)}<\lambda-\mu<\dfrac{1}{\pi(6-3\sqrt{3})}$ , then $z_3 \in (-\infty,-1)$ and $z_4 \in (-1,-\dfrac{1}{3});$

(iv) If $\lambda-\mu=\dfrac{1}{\pi(6-3\sqrt{3})},$ then $z_3 \in (-\infty,-1)$ and $z_4=-1,$ i.e. $z_4=-1$ is the threshold of $\sigma_{ess}(H_1)$;

(v) If $\lambda-\mu>\dfrac{1}{\pi(6-3\sqrt{3})}$, then $z_{3,}z_4 \in (-\infty,-1).$

We introduce the following sets:

$l_1=\{(\lambda,\mu):\lambda+\mu=0\},$

$l_2=\{(\lambda,\mu):\lambda-\mu=0\},$

$D_1=\{(\lambda,\mu):1/\pi(6-4\sqrt{3})\le\lambda+\mu<0\},$

$D_2=\{(\lambda,\mu):1/\pi(3\sqrt{3}-6)\le\lambda+\mu<1/\pi(6-4\sqrt{3})\},$

$D_3=\{(\lambda,\mu):0<\lambda-\mu\le1/\pi(4\sqrt{3}-6)\},$

$D_4=\{(\lambda,\mu):1/\pi(4\sqrt{3}-6)<\lambda-\mu<1/\pi(6-3\sqrt{3})\}.$

Combining Lemma 2.3 and Lemma 2.4 , we immediately obtain the following theorem.

**Theorem 2.5**

1. For $(\lambda,\mu)\in R^2\setminus(l_1\cup l_2)$ the operator $H$ has four eigenvalues. Moreover,

(i) if $(\lambda,\mu)\in(D_1\cap D_3),$ then all of the eigenvalues of the operator $H$ are in $\sigma_{ess}(H)$;

(ii) if $(\lambda,\mu)\in(D_2\cap D_3)\cup(D_1\cap D_4),$ then three of the eigenvalues of the operator $H$ are in $\sigma_{ess}(H)$;

(iii) if $(\lambda,\mu)\in D_3\setminus(D_1\cup D_2)\cup D_1\setminus(D_3\cup D_4)\cup(D_2\cap D_4),$ then two of the eigenvalues of the operator $H$ are in $\sigma_{ess}(H)$;

(iv) if $(\lambda,\mu)\in D_4\setminus(D_1\cup D_2)\cup D_2\setminus(D_3\cup D_4),$ then only one of the eigenvalues of the operator $H$ is in $\sigma_{ess}(H).$

2. For $(\lambda,\mu)\in(l_1\cup l_2)\setminus(0,0)$ the operator $H$ has two eigenvalues. Moreover,

(i) if $(\lambda,\mu)\in(D_1\cap l_2)\cup(D_3\cap l_1),$ then both of the eigenvalues of the operator $H$ are in $\sigma_{ess}(H)$;

(ii) if $(\lambda,\mu)\in(D_2\cap l_2)\cup(D_4\cap l_1),$ then only one of the eigenvalues of the operator $H$ is in $\sigma_{ess}(H).$

## *3.PROOF OF THE MAIN RESULTS.*

*Proof of the Lemma 2.1* . Consider a unitary operator $U$ acting in $L_2^{(2)}(\mathsf{T})$ as

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix},$$

where $I$ is a unit operator on $L_2(\mathsf{T})$. Then we verify following equality

$$HU^* = (H_0 - V)U^* = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{10} - \lambda\mathbf{v} - \mu\mathbf{v} & H_{20} - \lambda\mathbf{v} + \mu\mathbf{v} \\ H_{10} - \lambda\mathbf{v} - \mu\mathbf{v} & -H_{20} + \lambda\mathbf{v} - \mu\mathbf{v} \end{pmatrix}.$$

Therefore,

$$UHU^* = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \widetilde{H}.$$

*Proof of Lemma 2.2.* **Necessity**. Let a number $z$ be an eigenvalue of the operator $H_1$ corresponding to the eigenfunction $\varphi \in L_2(\mathsf{T})$, i.e

$$H_1\varphi = z\varphi, \quad \varphi \neq 0.$$

Then

$$(H_{10} - zI)\varphi = (\lambda + \mu)V\varphi$$

or

$$\varphi(x) = (\lambda + \mu)\frac{\cos xC_{1\varphi}}{E_{10}(x) - z} + (\lambda + \mu)\frac{\sin xC_{2\varphi}}{E_{10}(x) - z}, \quad (1)$$

since $E_{10}(x) \neq z$ for $z \in \mathsf{R} \setminus \sigma_{ess}(H_1)$, where

$$C_{1\varphi} = \int_{\mathsf{T}} \cos x\varphi(x)dx, \quad C_{2\varphi} = \int_{\mathsf{T}} \sin x\varphi(x)dx, \quad (2)$$

Substituting (1) to (2) we obtain

$$\begin{cases} C_{1\varphi}((\lambda + \mu)I_1(z) - 1) + (\lambda + \mu)K_1(z)C_{2\varphi} = 0, \\ (\lambda + \mu)K_1(z)C_{1\varphi} + C_{2\varphi}((\lambda + \mu)J_1(z) - 1) = 0. \end{cases}$$

Hence, we have

$$\begin{cases} C_{1\varphi}(1 - (\lambda + \mu)I_1(z)) = 0, \\ C_{2\varphi}(1 - (\lambda + \mu)J_1(z)) = 0, \end{cases} \quad (3)$$

since

$$K_1(z) = \int_{\mathsf{T}} \frac{\sin x \cos x dx}{-\frac{2}{3}\cos x - \frac{1}{3} - z} = 0.$$

Since $\varphi \neq 0$, so $(C_{1\varphi}, C_{2\varphi}) \neq (0,0)$. Hence from the equality (3) we obtain

$$\Delta_1(z) = (1 - (\lambda + \mu)I_1(z))(1 - (\lambda + \mu)J_2(z)) = 0, \quad z \notin \sigma_{ess}(H_1).$$

**Sufficiency.** Let $\Delta_1(z) = 0$ for $z \in \mathsf{R} \setminus \sigma_{ess}(H_1)$ and

$$\varphi(x) = (\lambda + \mu)\frac{\cos xC_{1\varphi} + \sin xC_{2\varphi}}{E_{10}(x) - z},$$

where $C_1, C_2$ are constants. Then, $\varphi \in L_2(\mathsf{T})$. Simple calculations show that

$$(H_1 - zI)\varphi = (\lambda + \mu)(C_1 \cos x(1 - (\lambda + \mu)I_1(z)) + C_2 \sin x(1 - (\lambda + \mu)J_1(z)).$$

If $(1 - (\lambda + \mu)I_1(z)) = 0$, then we put $C_1 = 0$ and $C_2 \neq 0$, and if $(1 - (\lambda + \mu)J_1(z)) = 0$, then we chose $C_1 \neq 0$ and $C_2 = 0$, since $\Delta_1(z) = (1 - (\lambda + \mu)I_1(z))(1 - (\lambda + \mu)J_1(z)) = 0$. Then we

have $(H_1 - zI)\varphi = 0$. Therefore, $\varphi$ is an eigenfunction of the operator $H_1$ corresponding to the eigenvalue $z$.

The proof of the theorem for the case $\alpha = 2$ can be proved similarly.

Using the classical identity

$$\int_{-\pi}^{\pi} \frac{dx}{A + B\cos x} = \frac{2\pi \operatorname{sgn}(A)}{\sqrt{A^2 - B^2}}, \quad |A| > |B| > 0 \qquad (4)$$

and making simple calculations we can represent the functions $I_\alpha(z)$ and $J_\alpha(z)$ into explicit forms as

$$(1)\ I_1(z) = \frac{9\pi}{2}\left(\left(\frac{1}{3} + z\right) - \frac{\operatorname{sgn}\left(\frac{1}{3} + z\right)\left(\frac{1}{3} + z\right)^2}{\sqrt{\left(\frac{1}{3} + z\right)^2 - \frac{4}{9}}}\right);$$

$$(2)\ J_1(z) = \frac{9\pi}{2}\left(\left(\operatorname{sgn}\left(\frac{1}{3} + z\right)\sqrt{\left(\frac{1}{3} + z\right)^2 - \frac{4}{9}}\right) - \left(\frac{1}{3} + z\right)\right);$$

$$(3)\ I_2(z) = -\frac{9\pi}{2}\left(\left(\frac{1}{3} - z\right) - \frac{\operatorname{sgn}\left(\frac{1}{3} - z\right)\left(\frac{1}{3} - z\right)^2}{\sqrt{\left(\frac{1}{3} - z\right)^2 - \frac{4}{9}}}\right);$$

$$(4)\ J_2(z) = -\frac{9\pi}{2}\left(\left(\operatorname{sgn}\left(\frac{1}{3} - z\right)\sqrt{\left(\frac{1}{3} - z\right)^2 - \frac{4}{9}}\right) - \left(\frac{1}{3} - z\right)\right).$$

*Proof of Lemma 2.3* . To prove lemma, according to the Lemma 2.2, we investigate zeros of the function

$$\Delta_1(z) = \Delta_1^e(z)\Delta_1^o(z)$$

and it is equivalent to the equations

$$\Delta_1^e(z) = 1 - (\lambda + \mu)I_1(z) = 0 \qquad (5)$$

and

$$\Delta_1^o(z) = 1 - (\lambda + \mu)J_1(z) = 0. \qquad (6)$$

(1) Let $\lambda + \mu > 0$. Then the functions $I_1(z)$ and $J_1(z)$ are positive on $(-\infty, -1)$ and strictly increasing, with range $(0, \infty)$. Hence, from the continuity $I_1(z)$ and $J_1(z)$ the equations $\Delta_1^e(z) = 0$ and $\Delta_1^o(z) = 0$ have unique solution in $(-\infty, -1)$.

Since $I_1(z) < 0$ and $J_1(z) < 0$ for $z \in (\frac{1}{3}, \infty)$, $\Delta_1^e(z) > 1$ and $\Delta_1^o(z) > 1$. Therefore, according to the Lemma 2.2 the operator $H_1$ has two eigenvalues $z_1, z_2$ and they lie on $(-\infty, -1)$.

(2) Let $\lambda + \mu < 0$. Then the function $I_1(z), J_1(z)$ are negative on $(\frac{1}{3}, \infty)$ and strictly increasing, with range $(-\infty, 0)$. Hence, from the continuity $I_1(z), J_1(z)$ the equations $\Delta_1^e(z) = 0$ and $\Delta_1^o(z) = 0$ have unique solution in $(\frac{1}{3}, \infty)$.

Since $I_1(z), J_1(z) > 0$ for $z \in (-\infty,-1)$, $\Delta_1^e(z) > 1$ and $\Delta_1^0(z) > 1$. Therefore, according to the Lemma 2.2 the operator $H_1$ has two eigenvalues , $z_1, z_2$ and they lie on $(\frac{1}{3},\infty)$.

(i) Let $\frac{1}{\pi(6-4\sqrt{3})} < \lambda + \mu < 0$. Note that

$$I_1(1) = \pi(6-4\sqrt{3}) \quad and \quad J_1(1) = \pi(3\sqrt{3}-6).$$

Then for $z > 1$

$\Delta_1^e(z) > \Delta_1^e(1) > 1 - \frac{1}{\pi(6-4\sqrt{3})} I_1(1) = 0$ and

$\Delta_1^o(z) > \Delta_1^o(1) > 1 - \frac{1}{\pi(6-4\sqrt{3})} J_1(1) > 1 - \frac{1}{\pi(3\sqrt{3}-6)} J_1(1) = 0$. According to (2) we can conclude that the roots of the both equations are in $(\frac{1}{3},1)$.

(ii) Let $\lambda + \mu = \frac{1}{\pi(6-4\sqrt{3})}$. One can see easily that the relation

$$z_1 = 1 \quad and \quad z_2 \in (\frac{1}{3},1)$$

for the roots of the equations respectively, $\Delta_1^e(z) = 0$ and $\Delta_1^o(z) = 0$ holds.

The rest of the proof of the lemma can be proved like above.

We omit the proof of the Lemma 2.4, since it proceeds analogously to the proof of the Lemma 2.3.

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