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Obuna indeksi – yakka tartibdagi obunachilar uchun **- 5583,**
tashkilot, korxonalar uchun **- 5584**

M U N D A R I J A
C O N T E N T S

№	Maqola nomi	Article title	Bet/ page
1	Kasr psevdoparabolik tenglama uchun Samarskiy-Ionkin ikki nuqtali chegaraviy shartlar **Sagdullaeva M.U., Xasanov G.A., Yuldashev T.K.**	Samarskii-Ionkin two-point boundary conditions for a fractional pseudoparabolic equation **Sagdullaeva M.U., Khasanov G.A., Yuldashev T.K.**	**5-14**
2	Gerasimov-Kaputo turidagi kasr tartibli nochiziqli differensial tenglamalar **Shukurova M.D., Xasanov G.A., Yuldashev T.K.**	Nonlinear differential equation with Gerasimov-Caputo type fractional operator and degenerator **Shukurova M.D., Khasanov G.A., Yuldashev T.K.**	**15-22**
3	NaH da lazer ta'sirida dissotsiatsiya dinamikasi **Umarov O., Jumabaev A.**	Laser-induced dissociation dynamics in NaH **Umarov O., Jumabaev A.**	**23-30**
4	Radon transformasidan foydalanish, tasvirni tanishda aylana sfera proyeksiyasini yaratish **Khatamov O., Mamanova B.**	Using the radon transform to create a projection of a circular sphere in image recognition **Khatamov O., Mamanova B.**	**31-38**
5	Flavonoid komplekslarining metal ionlari bilan sintezi, spektroskopik tadqiqotlar va supramolekulyar tizimlar **Ergashova Sh., Muxamadiyev N.**	Synthesis of flavonoid complexes with metal ions, spectroscopic studies and supramolecular systems **Ergashova Sh., Mukhamadiev N.**	**39-45**
6	2-benzil-6,7-dimetoksi-1,2,3,4-tetragidroizoxinolin hosilalari sintez reaksiyalarini o'rganish **Suyunova S.N., Quljonov J.X., Xudoyberdiyeva A.A., Urunbayeva Z.E. Z.E., Saidov A.Sh., Vinogradova V.I.**	Study of the reactions of synthesis of 2-benzyl-6,7-dimethoxy-1,2,3,4-tetrahydroisochinoline derivatives **Suyunova S.N., Kuljonov J.X., Khudoyberdieva A.A., Urunbaeva Z.E. Z.E., Saidov A.Sh., Vinogradova V.I.**	**46-50**
7	Keyingi qo'shni tugunlargacha o'zaro ta'sirlanuvchi ikki bozonli panjara gamiltoniani **Axmadova M., Axmatova Sh.**	A two-boson lattice hamiltonian with interactions up to next-neighboring sites **Akhmadova M., Akhmatova Sh.**	**51-58**
8	Sekin aylanishli magnitlangan neytron yulduzlarning plazma magnitosferasi o'zgartirilgan gravitatsiyada **Sayfiyev Sh.**	Plasma magnetosphere of slowly rotating magnetized neutron stars in modified gravity **Sayfiev Sh.**	**59-65**
9	Vaqtli qatorlar va mashinaviy o'qitish asosida sotuvdagi kechikishlarni aniqlash usullari **Nazarov F., Sayidqulov A.**	Methods for detecting delays in sales based on time series and machine learning **Nazarov F., Sayidkulov A.**	**66-71**
10	Integrallashgan matematik modellar va ularning qo'llanilishi **Safarov R., Xudoyberdiyev S.I.**	Integrated mathematical models and their applications **Safarov R., Khudoyberdiyev S.I.**	**72-77**
11	Chekli zichlikka va qator ko'rinishdagi moslangan manbaga ega Hirota tenglamasi uchun boshlang'ich qiymat masalasi **Eshbekov R.**	The initial value problem for the Hirota equation with finite density and self-consistent series source **Eshbekov R.**	**78-84**
12	L-sistein bilan modifikatsiyalangan mezog'ovak silikagel yordamida og'ir metall kationlarini ajratish **Ulugboyeva G.O., Kuliboyeva N.Sh., Uzokov J.R., Fazliyeva N.T., Muxamadiev N.Q.**	Removal of heavy metal cations using mesoporous silicagel modified with L-cysteine **Ulugboeva G.O., Kuliboeva N.Sh., Uzokov J.R., Fazlieva N.T., Mukhamadiev N.Q.**	**85-90**
13	Titan nano oksidi sintezi va uning fotokatalitik xossalarini zarracha o'lchamiga bog'liqligi **Mamaziyeva S., Avazova N., Tashmatova R., Muxamadiev N., Muxamadiev A., Uzokov J.**	Synthesis of nano titanium oxide and dependence of its photocatalytic properties on particle size **Mamazieva S., Avazova N., Tashmatova R., Mukhamadiev N., Mukhamadiev A., Uzokov J.**	**91-96**
14	$SrTiO_3$ nanozarrachalarini sintez qilish va ularning o'lcham-fotokatalitik xossalari **Hayitova M., Baxranova M.A., Muxamadiyev N.Q.**	Synthesis of $SrTiO_3$ and its size-dependent photocatalytic properties **Hayitova M., Bakhranova M.A., Mukhamadiyev N.Q.**	**97-102**

№	Maqola nomi	Article title	Bet/ page
15	Rekreatsiya maqsadlarida tabiatdan foydalanishni boshqarish **Rafikova N.**	Management of nature utilization for recreational purposes **Rafikova N.**	**103-110**
16	Modifikatsiyalangan faollashgan ko‘mir yordamida eritma tarkibidagi organik bo‘yoqlarni ajratish **Ergasheva D., Muxamadiyev A., Muxamadiyev N., Uzokov J.**	Separation of organic dyes from solutions using modified activated carbon **Ergasheva D., Mukhamadiyev A., Mukhamadiyev N., Uzokov J.**	**111-116**
17	Neyron tarmoqli modellar asosida qandli diabetni tashxislashning intellektual algoritmlari **Sariyev Sh.N.**	Intelligent algorithms for diagnosing diabetes mellitus based on neural network models **Sariev Sh.N.**	**117-123**
18	Mahalliy kaolindan seolitlar sintez va fizik-kimyoviy tahlili **Magdiev Sh., Majidova F., Nasimov A., Tashpulatov X., Toshpulatov D., Mirzayev Sh., Norboev K., Normuradov O.**	Synthesis and physicochemical analysis of zeolite from local kaolin **Magdiev Sh., Majidova F., Nasimov A., Tashpulatov Kh., Toshpulatov D., Mirzaev Sh., Norboev K., Normuradov O.**	**124-129**
19	Densenet modelini transfer o‘qitish asosida masofadan zondlash sahnalarini tasniflash aniqligini oshirish maqolaning o‘zbek tilidagi nomi **Yusupov O.R., Eshonqulov E.Sh.**	Enhancing the accuracy of remote sensing scene classification using transfer learning with densenet **Yusupov O.R., Eshonkulov E.Sh.**	**130-144**
20	Neft va gaz quduqlari minerallashgan qatlam suvlarining kimyoviy tarkibi va fizik-kimyoviy xususiyatlarini o‘rganish **Kenjaev N.A., Igamkulova N.A.**	Chemical composition and physico-chemical properties of mineralized formation waters from oil and gas wells **Kenjaev N.A., Igamkulova N.A.**	**145-148**

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SAMARSKII-IONKIN TWO-POINT BOUNDARY CONDITIONS FOR A FRACTIONAL PSEUDOPARABOLIC EQUATION

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Abstract. This paper is devoted to study the mixed problem for a nonhomogeneous pseudoparabolic equation. The Hilfer fractional equation is studied with initial value conditions recording first variable t and Samarskii-Ionkin type boundary value conditions recording second variable x. Spectral problem is studied and eigenvalues, eigenfunctions are found. The solution is constructed on the form of Fourier series. Existence and uniqueness of solution's theorems are proved.

Keywords: Mixed problem, Hilfer type fractional equation, Samarskii-Ionkin type conditions, eigenvalues, eigenfunctions, Fourier series, existence and uniqueness theorems.

Kasr psevdoparabolik tenglama uchun Samarskiy-Ionkin ikki nuqtali chegaraviy shartlar

Annotatsiya. Ushbu maqola bir jinsli bo‘lmagan psevdoparabolik tenglama uchun aralash muammoni o‘rganishga bag'ishlangan. Hilfer kasr tartibli tenglama birinchi o‘zgaruvchini qayd qiluvchi boshlang‘ich qiymat shartlari va ikkinchi o‘zgaruvchini qayd qiluvchi Samarskiy-Ionkin tipidagi chegaraviy qiymat shartlari bilan o‘rganiladi. Spektral masala o‘rganilib, xos qiymatlar, xos funksiyalar topiladi. Yechim Furye seriyasi shaklida qurilgan. Yechim mavjudligi va yagonaligi haqidagi teoremalar isbotlangan.

Kalit so‘zlar: Aralash masala, Hilfer kasr tartibli tenglama, Samarskiy-Ionkin turidagi shartlar, xos qiymatlar, xos funksiyalar, Furye qatori, mavjudlik va yagonalik teoremalari.

Двухточечные краевые условия Самарского-Ионкина для дробного псевдопараболического уравнения

Аннотация. Работа посвящена исследованию смешанной задачи для неоднородного псевдопараболического уравнения. Дробное уравнение Хилфера изучается с начальными условиями, содержащими первую переменную, и краевыми условиями типа Самарского-Ионкина, содержащими вторую переменную. Изучается спектральная задача и находятся собственные значения, собственные функции. Решение строится в виде ряда Фурье. Доказываются теоремы о существовании и единственности решения.

Ключевые слова: смешанная задача, дробное уравнение Хилфера, условия типа Самарского-Ионкина, собственные значения, собственные функции, ряды Фурье, теоремы существования и единственности.

Introduction. Problem statement. Partial functional differential equations, containing a fractional integro-differential operator, arise when solving different kind problems of nonlinear mechanics and nonlinear optimal control, biology and so on [1-6]. So, theoretical interest arisen in studying nonlocal problems for fractional functional differential equations. By today a lot of publications of studying nonlocal problems for fractional differential equations, describing many natural and practical processes, are appeared (see, for example, [7-11]). The interest on the study of nonlocal problems for the fractional differential equations is only increasing by the mathematicians.

In the domain $\Omega \equiv (t_0, T) \times (0,1)$ we consider the equation

$$\left(D_{t_0 t}^{\alpha,\gamma} + D_{t_0 t}^{\alpha,\gamma} \frac{\partial^5}{\partial x^4} + \frac{\partial^4}{\partial x^4} \right) U(t,x) = f(t,x) \tag{1}$$

with initial value conditions

$$\lim_{t \to +t_0} J_{t_0 t}^{2-\gamma} U(t,x) = \varphi(x), \quad \lim_{t \to +t_0} \frac{\partial}{\partial t} J_{t_0 t}^{2-\gamma} U(t,x) = \psi(x), \tag{2}$$

$$U(t,1) = 0, \quad U_{xx}(t,0) = 0, \tag{3}$$

$$U_x(t,0) = U_x(t,1), \; U_{xxx}(t,0) = U_{xxx}(t,1), \; 0 \le t \le T, \tag{4}$$

where $\varphi(x), \psi(x), f(t,x)$ are given functions, $D_{t_0 t}^{\alpha,\gamma}$ is Hilfer fractional operator,

$$D_{t_0 t}^{\alpha,\gamma} = J_{t_0 t}^{\gamma-\alpha} \frac{d^2}{dt^2} J_{t_0 t}^{2-\gamma}, \quad 1 < \alpha \le 2, \quad \alpha < \gamma \le 2,$$

$J_{t_0 t}^{\alpha} \eta(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} \eta(s)\, ds$ is Riemann-Liouville integral operator, $0 \le t_0 < T$.

Problem. *We find a function* $U(t,x)$*, that satisfies the fractional differential equation* (1)*, the initial conditions* (2)*, the boundary conditions* (3), (4) *and belongs to the class of functions*

$$(t-t_0)^{2-\gamma} U(t,x) \in C_{t,x}^{1,3}(\overline{\Omega}), \quad D_{0t}^{\alpha,\gamma} U_{xxxx}(t,x) \in C(\Omega), \quad U_{xxxx}(t,x) \in C(\Omega). \tag{5}$$

Some auxiliary materials about riesz bases. Let $\{\vartheta_n(x)\}$ and $\{\omega_n(x)\}$ be two complete systems of functions from $L_2[a,b]$. Let us by $(\vartheta, \omega)_0$ denote the scalar product of functions $\vartheta(x)$ and $\omega(x)$ in $L_2[a,b]$:

$$(\vartheta, \omega)_0 = (\vartheta, \omega)_{L_2[a,b]} = \int_a^b \vartheta(x) \omega(x) dx.$$

Definition 1 [12]. *Systems* $\{\vartheta_n(x)\}$ *and* $\{\omega_n(x)\}$ *form a biorthonormal system on some interval* $[a,b]$*, if*

$$(\vartheta_n, \omega_k)_0 = \int_a^b \vartheta_n \omega_k dx = \delta_{nk} = \begin{cases} 0, & n \ne k, \\ 1, & n = k, \end{cases}$$

in this case the system $\{\omega_n(x)\}$ *is called biorthogonally adjoint to the system* $\{\vartheta_n(x)\}$ *on the interval* $[a,b]$.

Definition 2 [12]. *A system of elements* $\{\vartheta_n(x)\}$ *is called minimal in* $L_2[a,b]$*, if none of its elements belongs to the closure of the linear span of the other elements of this system.*

The minimality of the system ensures the existence of a biorthogonally conjugate system.

Definition 3 [12]. *The biorthogonal expansion of a function* $f \in L_2[a,b]$ *in a system* $\{\vartheta_n(x)\}$ *is called a series*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \vartheta_n(x),$$

where $c_n = (f, \omega_n)_0$.

Definition 4 [12]. *A complete and minimal system of functions* $\{\vartheta_n(x)\}$ *is called Besselian if for any* $f \in L_2[a,b]$ *the series of squares of the coefficients of its biorthogonal expansion in* $\{\vartheta_n(x)\}$ *converges, i.e. if it follows from any function* $f \in L_2[a,b]$ *that*

$$\sum_{n=1}^{\infty} |(f,\omega_n)_0|^2 < \infty,$$

where $\{\omega_n\}$ *is biorthogonally conjugate system to* $\{\vartheta_n(x)\}$.

Definition 5 [12]. *A complete and minimal system of functions* $\{\vartheta_n(x)\}$ *is called a Hilbert system, if for any sequence of numbers* c_n, *such that* $\sum_{k=1}^{\infty} c_n^2 < \infty$, *there is one and only one function* $f \in L_2[a,b]$, *for which* c_n *are coefficients of its biorthogonal expansion in* $\{\vartheta_n(x)\}$:

$$c_n = (f,\omega_n)_0, \quad n = 1,2,...$$

Definition 6 [12]. *A complete and minimal system is called a Riesz basis, if it is both Bessel and Hilbert basis.*

Theorem 1 [13]. *The following statements are equivalent:*

1) *The sequence of functions* $\{\omega_j(x)\}_1^{\infty}$ *forms a basis, equivalent to the orthonormal one, in the space* R;

2) *The sequence of functions* $\{\omega_j(x)\}_1^{\infty}$ *will be an orthonormal basis of the space R with the appropriate replacement of the scalar product* (f,g) *by some new one* $(f,g)_1$, *topologically equivalent to the original one;*

3) *The sequence of functions* $\{\omega_j(x)\}_1^{\infty}$ *is complete in R and there exist constants,* a_1, $a_2 (>0)$, *such that for any natural* n *and any complex numbers* $\gamma_1, \gamma_2, ..., \gamma_n$ *there holds*

$$a_2 \sum_{j=1}^{n} |\gamma_j|^2 \le \sum_{j=1}^{n} |\gamma_j \omega_j|^2 \le a_1 \sum_{j=1}^{n} |\gamma_j|^2;$$

4) *The sequence of functions* $\{\omega_j(x)\}_1^{\infty}$ *is complete in R and its Gram matrix* $(\omega_j(x), \omega_k(x))_1^{\infty}$ *generates a bounded invertible operator in the space* ℓ_2;

5) *The sequence of functions* $\{\omega_j(x)\}_1^{\infty}$ *is complete in R, corresponds to complete biorthogonal sequence of functions* $\{\chi_j(x)\}_1^{\infty}$ *and for any* $f(x) \in R$ *it is true that*

$$\sum_{j=1}^{n} |(f,\omega_j)|^2 < \infty, \quad \sum_{j=1}^{n} |(f,\chi_j)|^2 < \infty.$$

Lemma 1 [14]. *Let be* $f(x) \in L_2[0,1]$ *and* $a_k = \int_0^1 f(x) e^{-\lambda k x} dx, \quad b_k = \int_0^1 f(x) e^{\lambda k (x-1)} dx,$ *where* λ *is an arbitrary complex number with a positive real part* ($\lambda = \alpha + i\beta, \ \alpha > 0$). *Then the series* $\sum_{k=1}^{\infty} |a_k|^2, \quad \sum_{k=1}^{\infty} |b_k|^2$ *are convergence.*

Eigenvalues and eigenfunction. Sigenvalues and eigenfunctions. First, we consider homogeneous differential equation

$$\frac{\partial U(t,x)}{\partial t} + \frac{\partial^5 U(t,x)}{\partial t \partial x^4} + \frac{\partial^4 U(t,x)}{\partial x^4} = 0. \tag{6}$$

We will look for a non-trivial particular solution of the equation in the form $U(t,x)=u(t)\cdot\vartheta(x).$ Substituting this product of functions, depending from different variables, into equation (6), we obtain

$$-\frac{u'(t)}{u'(t)+u(t)}=\frac{\vartheta^{(IV)}(x)}{\vartheta(x)}.$$

Hence, equating second fraction into λ we obtain

$$\vartheta^{(IV)}(x)-\lambda\vartheta(x)=0,\quad \lambda\geq 0. \tag{7}$$

Using conditions (3) and (4), from product of two functions we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(1)=0,\quad \vartheta''(0)=0,\quad \vartheta'(0)=\vartheta'(1),\quad \vartheta'''(0)=\vartheta'''(1). \tag{8}$$

Solving the spectral problem (7), (8), we derive the eigenvalues

$$\lambda_n=(2\pi n)^4,\quad n=0,1,2,... \tag{9}$$

Eigenfunctions, corresponding to the eigenvalues (9), have the forms

$$\vartheta_0(x)=2(1-x),\quad \vartheta_{1n}(x)=-2\sin 2\pi nx,\quad \vartheta_{2n}(x)=\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx. \tag{10}$$

The spectral problem (7), (8) is not self-adjoint and it is easy to see that the following problem will be adjoint to it

$$\omega^{(IV)}(x)-\lambda\omega(x)=0,\quad 0<x<1, \tag{11}$$

$$\omega(0)=\omega(1),\quad \omega'(1)=0,\quad \omega''(0)=\omega''(1),\quad \omega'''(0)=0. \tag{12}$$

Along with problem (7), (8), we also consider adjoint to it problem (11), (12). Solving this problem, it is not difficult to see that the eigenfunctions, corresponding to eigenvalues (9), have the form

$$\omega_0(x)=1,\quad \omega_{1n}(x)=\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx,\quad \omega_{2n}(x)=-2\cos 2\pi nx. \tag{13}$$

It should be noted that (10) and (13) are a non-orthogonal system of functions. Indeed, let us consider, for example, The system (10) and calculate it

$$\left(\vartheta_0(x),\vartheta_n^{(1)}(x)\right)_0=-4\int_0^1(1-x)\sin 2\pi nxdx=-\frac{2}{\pi n}\neq 0.$$

Let us move on to studying the issues of the basis of systems (10) and (13) in $L_2[0,1]$.

Lemma 2. *Systems of functions* (10) *and* (13) *are biorthogonal systems in* $L_2[0,1]$:

$$(\vartheta_0,\omega_0)_0=1,\quad (\vartheta_{ik},\omega_{jn})_0=\begin{cases}1, & k=n,\ i=j,\\ 0, & k\neq n,\ i\neq j,\end{cases}\quad i,j=1,2,\quad n,k=1,2,...$$

We present the proof of Lemma 2 for the functions $\vartheta_{1n}(x)$ and $\omega_{1n}(x)$. According to Definition 1, we calculate the integral

$$(\vartheta_{1k},\omega_{1n})_0=-2\int_0^1\sin 2\pi kx\left(\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx\right)dx=$$

$$-\frac{2}{e^{2\pi n}-1}\int_0^1(e^{2\pi nx}+e^{2\pi n(1-x)})\sin 2\pi kxdx+2\int_0^1\sin 2\pi nx\sin 2\pi kxdx=I_{kn}+J_{kn}.$$

Simple calculations are showed that

$$I_{kn}=-\frac{2}{e^{2\pi n}-1}\int_0^1(e^{2\pi nx}+e^{2\pi n(1-x)})\sin 2\pi kxdx=0,\quad k,n\in N,$$

$$J_{kn}=2\int_0^1\sin 2\pi kx\cdot\sin 2\pi nxdx=\begin{cases}1, & k=n,\\ 0, & k\neq n,\end{cases}\quad k,n\in N.$$

Consequently, $(\vartheta_{1n},\omega_{1n})_0=1$ for $k=n$ and $(\vartheta_{1n},\omega_{1n})_0=0$ for $k\neq n.$ The Lemma 2 is proved.

Lemma 3. *The systems of functions* (10) *and* (13) *are minimal in* $L_2[0,1]$.

The *proof* of Lemma 3 follows from the existence of a biorthonormal system, which was established in Lemma 2.

Theorem 2. *The system of functions* (10) *and* (13) *is complete in space* $L_2[0,1]$.

Proof. First, we prove the completeness of (10). Assume the opposite, let the system of functions (10) be incomplete in $L_2[0,1]$. Then there exists a function $\psi(x)$ from $L_2[0,1]$, that is orthogonal to all functions of the system (10). We will expand the function $\psi(x)$ into a Fourier series

$$\psi(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx),$$

which is convergence in $L_2[0,1]$. Since $\psi(x)$ is orthogonal to the system $\{-2\sin 2\pi nx\}_{n=1}^{\infty}$, the last expansion will take the form

$$\psi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx. \tag{14}$$

By assumption, $\psi(x)$ is orthogonal to all functions of the form $\vartheta_0(x)$, $\vartheta_{2k}(x)$. Next, multiplying the series (14) successively by these functions and integrating over the interval $(0,1)$, we have

$$0 = 2\int_0^1 \psi(x)(1-x)dx = 2a_0\int_0^1 (1-x)dx + 2\sum_{n=1}^{\infty} a_n \int_0^1 (1-x)\cos 2\pi nx dx = a_0,$$

$$0 = \int_0^1 \psi(x)\left(\frac{e^{2\pi kx} - e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\right)dx =$$

$$= \sum_{n=1}^{\infty} a_n \int_0^1 \left(\frac{e^{2\pi kx} - e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \cos 2\pi kx\right)\cos 2\pi nx dx = -\frac{1}{2}a_k, \quad k = 1,2,3,...$$

Hence, follows that $a_k = 0, \quad k = 0,1,2,...$ Therefore, from (14) we obtain that $\psi(x) = 0$ on $[0,1]$, which contradicts the condition $\psi(x) \neq 0$. Thus, the system (10) is complete in the space $L_2[0,1]$.

Now we will prove the completeness of the system (13). Let there $\psi(x)$ be a function from $L_2[0,1]$, different from zero, orthogonal to all functions of system (13). Since the function $\psi(x)$ is orthogonal to system $\{-2\cos 2\pi nx\}_{n=0}^{\infty}$, it can be represented in $L_2[0,1]$ as a sine series

$$\psi(x) = \sum_{n=1}^{\infty} b_n \sin 2\pi nx. \tag{15}$$

Next, multiplying the last series by $\omega_{1k}(x)$ and integrating over the segment $[0,1]$, taking into account the orthogonality of the functions $\psi(x)$ and $\omega_{1k}(x)$, we obtain

$$0 = \int_0^1 \psi(x)\left(\frac{e^{2\pi kx} + e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \sin 2\pi kx\right)dx =$$

$$= \sum_{n=0}^{\infty} b_n \int_0^1 \left(\frac{e^{2\pi kx} + e^{2\pi k(1-x)}}{e^{2\pi k}-1} - \sin 2\pi kx\right)\sin 2\pi nx dx = -\frac{1}{2}b_k, \quad k = 1,2,...,$$

i.,e. $b_k = 0, \quad n = 1,2,...$ Then from (15) it follows that $\psi(x) = 0$ on $[0,1]$, i.e. system (13) is complete in $L_2[0,1]$. Theorem 2 is proved.

Theorem 3. *The system of functions* (10) *and* (13) *forms Riesz basis in* $L_2[0,1]$.

To prove the Riesz basis property of the systems (10) and (13), according to Theorem 1, it is sufficient to establish the completeness of these systems and the convergence for any $\psi(x)\in L_2[0,1]$ of the following series:

$$(\psi(x),2(1-x))_0^2+\sum_{n=1}^{\infty}(\psi(x),-2\sin 2\pi nx)_0^2+$$
$$+\sum_{n=1}^{\infty}\left(\psi(x),\frac{1}{e^{2\pi n}-1}(e^{2\pi nx}-e^{2\pi n(1-x)})-\cos 2\pi nx\right)_0^2, \quad (16)$$
$$(\psi(x),1)_0^2+\sum_{n=1}^{\infty}(\psi(x),-2\cos 2\pi nx)_0^2+$$
$$+\sum_{n=1}^{\infty}\left(\frac{1}{e^{2\pi n}-1}(e^{2\pi nx}+e^{2\pi n(1-x)})-\sin 2\pi nx,\psi(x)\right)_0^2. \quad (17)$$

The completeness of systems (10) and (13) follows from Lemma 2 and therefore we will show the convergence of series (16) and (17). Let us consider the series (11) and use the notation

$$I_1=4(\psi(x),(1-x))_0^2,\quad I_2=4\sum_{n=1}^{\infty}(\psi(x),\sin 2\pi nx)_0^2,$$
$$I_3=\sum_{n=1}^{\infty}\left(\psi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2.$$

Applying the Cauchy-Shwartz inequality, for I_1 we obtain

$$I_1=4\left(\int_0^1(1-x)\psi(x)dx\right)^2\le 4\int_0^1(1-x)^2dx\int_0^1\psi^2(x)dx=\frac{4}{3}\|\psi(x)\|^2_{L_2(0,1)}<\infty.$$

Analogously, we have

$$I_2=4\sum_{n=1}^{\infty}(\psi(x),\sin 2\pi nx)_0^2=2\sum_{n=1}^{\infty}\left(\psi(x),\sqrt{2}\sin 2\pi nx\right)^2=2\sum_{n=1}^{\infty}c_n^2,$$

where $c_n=\left(\psi(x),\sqrt{2}\sin 2\pi nx\right)$ are Fourier coefficients of the function $\psi(x)$ in an orthonormal system $\{\sqrt{2}\sin 2\pi nx\}$. Hence, applying Bessel's inequality, we obtain, that

$$I_2=2\sum_{n=1}^{\infty}c_n^2\le 2\|\psi(x)\|^2_{L_2[0,1]}<\infty.$$

Next, we consider I_3. Since

$$A=\left(\psi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2=$$
$$=\left((\psi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1})-(\psi(x),\cos 2\pi nx)\right)_0^2,$$

applying the inequality $(a+b)^2\le 2(a^2+b^2),$ we obtain

$$A\le 2\left(\psi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}\right)_0^2+2(\psi(x),\cos 2\pi nx)_0^2=$$
$$=2\left(\left(\psi(x),\frac{e^{2\pi nx}}{e^{2\pi n}-1}\right)-\left(\psi(x),\frac{e^{2\pi n(1-x)}}{e^{2\pi n}-1}\right)\right)_0^2+2(\psi(x),\cos 2\pi nx)_0^2.$$

Applying the previous inequality again, we get that

$$A\le 4\left(\psi(x),\frac{e^{2\pi nx}}{e^{2\pi n}-1}\right)_0^2+4\left(\psi(x),\frac{e^{2\pi n(1-x)}}{e^{2\pi n}-1}\right)_0^2+2(\psi(x),\cos 2\pi nx)_0^2.$$

Hence, we obtain

$$I_3\le 4\sum_{n=1}^{\infty}\left(\psi(x),\frac{e^{2\pi nx}}{e^{2\pi n}-1}\right)_0^2+4\sum_{n=1}^{\infty}\left(\psi(x),\frac{e^{2\pi n(1-x)}}{e^{2\pi n}-1}\right)_0^2+2\sum_{n=1}^{\infty}(\psi(x),\cos 2\pi nx)_0^2=$$
$$=J_1+J_2+J_3.$$

For the J_3 we have

$$J_3 = 2\sum_{n=1}^{\infty}\left(\psi(x), \cos 2\pi nx\right)^2 = \sum_{n=1}^{\infty} a_n^2,$$

where $a_n = \left(\psi(x), \sqrt{2}\cos 2\pi nx\right)$ are Fourier coefficients for function $\psi(x)$ in orthonormal system $\{\sqrt{2}\cos 2\pi nx\}$. Then, applying Bessel's inequality, we obtain that

$$J_3 = \sum_{n=1}^{\infty} a_n^2 \le \left\|\psi(x)\right\|^2_{L_2[0,1]} < \infty.$$

Since

$$\left(\psi(x), \frac{e^{2\pi nx}}{e^{2\pi n}-1}\right)_0^2 = \left(\int_0^1 \psi(x)\cdot\frac{e^{2\pi n}}{e^{2\pi n}-1}e^{2\pi n(x-1)}dx\right)_0^2 =$$

$$= \left(\int_0^1 \psi(x)\left[1+\frac{1}{e^{2\pi n}-1}\right]e^{2\pi n(x-1)}dx\right)^2 \le 4\left(\int_0^1 \psi(x)e^{2\pi n(x-1)}dx\right)^2,$$

for J_1 we derive

$$J_1 \le 16\sum_{n=1}^{\infty}\left(\int_0^1 \psi(x)e^{2\pi n(x-1)}dx\right)^2 = 16\sum_{n=1}^{\infty} b_n^2, \quad b_n = \int_0^1 \psi(x)e^{2\pi n(1-x)}dx.$$

Hence, taking into account Lemma 1, we obtain that J_1 is finite. Similarly, we obtain that J_2 is also finite. Thus, the series I_1 and I_2 converge. Therefore, the series (16) also converges. The convergence of the series (17) is proved similarly. Theorem 3 is proved.

Construction of the solutions of the problem (1)-(4). Taking into account the formulas (10) and (13) we look for a solution to the problem (1)-(4) in the form of following Fourier series:

$$U(t,x) = u_0(t)\vartheta_0(x) + \sum_{n=1}^{\infty}\left(u_{1,n}(t)\vartheta_{1,n}(x) + u_{2,n}(t)\vartheta_{2,n}(x)\right), \tag{18}$$

where

$$u_0(t) = \int_0^1 U(t,y)\omega_0(y)dy, \quad u_{1,n}(t) = \int_0^1 U(t,y)\omega_{1,n}(y)dy, \tag{19}$$

$$u_{2,n}(t) = \int_0^1 U(t,y)\omega_{2,n}(y)dy. \tag{20}$$

Let the function $U(t,x)$ be a solution to the mixed problem (1)-(4). Then, substituting representation (18) into equation (1) and taking (10) and (13) into account, we obtain

$$D_{t_0t}^{\alpha,\gamma}u_0(t)\vartheta_0(x) + \sum_{n=1}^{\infty}\left(D_{t_0t}^{\alpha,\gamma}u_{1,n}(t)\vartheta_{1,n}(x) + D_{t_0t}^{\alpha,\gamma}u_{2,n}(t)\vartheta_{2,n}(x)\right) +$$

$$+\sum_{n=1}^{\infty}\lambda_n\left[D_{t_0t}^{\alpha,\gamma}u_{1,n}(t)\vartheta_{1,n}(x) + D_{t_0t}^{\alpha,\gamma}u_{2,n}(t)\vartheta_{2,n}(x)\right] + \sum_{n=1}^{\infty}\lambda_n\left[u_{1,n}(t)\vartheta_{1,n}(x) + u_{2,n}(t)\vartheta_{2,n}(x)\right] +$$

$$= f_0(t)\vartheta_0(x) + \sum_{n=1}^{\infty}\left(f_{1,n}(t)\vartheta_{1,n}(x) + f_{2,n}(t)\vartheta_{2,n}(x)\right),$$

where

$$f_0(t) = \int_0^1 f(t,y)\omega_0(y)dy, \quad f_{1,n}(t) = \int_0^1 f(t,y)\omega_{1,n}(y)dy, \tag{21}$$

$$f_{2,n}(t) = \int_0^1 f(t,y)\omega_{2,n}(y)dy. \tag{22}$$

Hence, by virtue of (19)-(22) into account, we obtain

$$D_{t_0t}^{\alpha,\gamma}u_0(t) = f_0(t), \tag{23}$$

$$D_{t_0t}^{\alpha,\gamma}u_{\kappa,n}(t)+\mu_n u_{\kappa,n}(t)=\frac{f_{\kappa,n}(t)}{1+\lambda_n},\quad \kappa=1,2, \tag{24}$$

where

$$\mu_n=\frac{\lambda_n}{1+\lambda_n},\quad \lambda_n=(2n\pi)^4.$$

Taking into account the formulas (10) and (13), we consider the function $\varphi(x)$ as in the case of (18):

$$\varphi(x)=\varphi_0\,\vartheta_0(x)+\sum_{n=1}^{\infty}\left(\varphi_{1,n}\,\vartheta_{1,n}(x)+\varphi_{2,n}\,\vartheta_{2,n}(x)\right),$$
$$\psi(x)=\psi_0\,\vartheta_0(x)+\sum_{n=1}^{\infty}\left(\psi_{1,n}\,\vartheta_{1,n}(x)+\psi_{2,n}\,\vartheta_{2,n}(x)\right), \tag{25}$$

where

$$\varphi_0(t)=\int_0^1\varphi(y)\omega_0(y)dy,\quad \varphi_{\kappa,n}(t)=\int_0^1\varphi(y)\omega_{\kappa,n}(y)dy, \tag{26}$$

$$\psi_0(t)=\int_0^1\psi(y)\omega_0(y)dy,\quad \psi_{\kappa,n}(t)=\int_0^1\psi(t,y)\omega_{\kappa,n}(y)dy,\ \kappa=1,2. \tag{27}$$

The differential equation (23) is scalar equation. The differential equation (24) is the countable system of differential equations. Taking (25) into account, from the two-point condition (2) we obtain

$$\lim_{t\to+t_0}J_{t_0t}^{2-\gamma}u_0(t)=\int_0^1\lim_{t\to+t_0}J_{t_0t}^{2-\gamma}U(t,y)\omega_0(y)dy=\int_0^1\varphi(x)\omega_0(y)dy=\varphi_0, \tag{26}$$

$$\lim_{t\to+t_0}\frac{\partial}{\partial t}u_{\kappa,n}(t)=\int_0^1\lim_{t\to+t_0}J_{t_0t}^{2-\gamma}\frac{\partial}{\partial t}U_\kappa(t,y)\omega_{\kappa,n}(y)dy=\int_0^1\psi(x)\omega_{\kappa,n}(y)dy=\psi_{\kappa,n},\quad \kappa=1,2. \tag{27}$$

First, we solve the equation (23). There holds the following lemma.

Lemma 4 ([15])**.** *The solution of the differential equation* (23) *with initial value conditions* (26) *and* (27) *is represented as follows*

$$u_0(t)=\varphi_0(t-t_0)^{\gamma-2}+\psi_0(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f_0(s)ds. \tag{28}$$

Now we solve the countable system (24) of ordinary differential equations with conditions (26) and (27).

Lemma 5 ([15])**.** *The solution of the differential equation* (24) *with initial value conditions* (26) *and* (27) *is represented as follows*

$$u_{\kappa,n}(t)=G_n(t)+$$
$$+\frac{1}{1-\sigma_2}\int_{t_0}^{t}s(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n\cdot(t-s)^\alpha\right)\int_{t_0}^{T}\varsigma\int_{t_0}^{\varsigma}(\varsigma-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(\varsigma-\xi)^\alpha\right)f_{\kappa,n}(s)\,d\xi\,d\varsigma\,ds+$$
$$+\int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^\alpha\right)f_{\kappa,n}(s)\,ds, \tag{29}$$

where

$$G_n(t)=P_n(t)+\frac{\sigma_{1,n}}{1-\sigma_{2,n}}\int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^\alpha\right)\cdot s\,ds,$$
$$\sigma_{1,n}=\int_{t_0}^{T}s\cdot P_n(s)\,ds,\quad \sigma_{2,n}=\int_{t_0}^{T}s\int_{t_0}^{s}\xi\cdot(s-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(s-\xi)^\alpha\right)d\xi\,ds\neq1,$$
$$P_n(t)=(t-t_0)^{\gamma-2}\varphi_nE_{\alpha,\gamma-1}\left(-\mu_n(t-t_0)^\alpha\right)+(t-t_0)^{\gamma-1}\psi_nE_{\alpha,\gamma}\left(-\mu_n(t-t_0)^\alpha\right),$$
$$\mu_n=\frac{\lambda_n}{1+\lambda_n},\quad \lambda_n=(2n\pi)^4.$$

Taking into account the representations (28) and (29), the series (18) we write as

$$U(t,x)=\vartheta_0(x)\left[\varphi_0(t-t_0)^{\gamma-2}+\psi_0(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f_0(s)ds\right]+$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\vartheta_{\kappa,n}(x)\left[G_n(t)+\int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^{\alpha}\right)f_{\kappa,n}(s)\,ds+\right.$$

$$\left.+\frac{1}{1-\sigma_2}\int_{t_0}^{t}s(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^{\alpha}\right)\int_{t_0}^{T}\varsigma\int_{t_0}^{\varsigma}(\varsigma-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(\varsigma-\xi)^{\alpha}\right)f_{\kappa,n}(s)\,d\xi\,d\varsigma\,ds\right]. \quad (30)$$

The series (30) has singularity at the point $t=t_0$, so we multiple it to $(t-t_0)^{2-\gamma}$:

$$(t-t_0)^{2-\gamma}U(t,x)=\vartheta_0(x)\left[\varphi_0+\psi_0(t-t_0)+\frac{(t-t_0)^{2-\gamma}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f_0(s)ds\right]+$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\vartheta_{\kappa,n}(x)\left[(t-t_0)^{2-\gamma}G_n(t)+(t-t_0)^{2-\gamma}\int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^{\alpha}\right)f_{\kappa,n}(s)\,ds+\right.$$

$$\left.+\frac{(t-t_0)^{2-\gamma}}{1-\sigma_2}\int_{t_0}^{t}s(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(t-s)^{\alpha}\right)\int_{t_0}^{T}\varsigma\int_{t_0}^{\varsigma}(\varsigma-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(-\mu_n(\varsigma-\xi)^{\alpha}\right)f_{\kappa,n}(s)\,d\xi\,d\varsigma\,ds\right]. \quad (31)$$

Unique solvability of the problem (1)-(4). Smoothness condition S**.** Let the functions $\varphi(x),\psi(x)$ and $f(t,x)$ are satisfying the conditions

$$\varphi(x)\in C^5[0,1],\quad \varphi(1)=\varphi''(0)=\varphi^{(IV)}(1)=0,\quad \varphi'(0)=\varphi'(1),\quad \varphi'''(0)=\varphi'''(1),$$

$$\psi(x)\in C^5[0,1],\quad f(t,x)\in C^{0,1}_{t,x}(\bar{\Omega}).$$

Then, we integrate by parts (19), (20) and (25), (26) five times and (21), (22) one times on the variable x, respectively, and obtain

$$\varphi_{1,n}=-\left(\frac{1}{2\pi}\right)^5\frac{\varphi^{(V)}_{1,n}}{n^5},\quad \varphi^{(V)}_{1,n}=\int_0^1\frac{\partial^5\varphi(y)}{\partial y^5}\left(\frac{e^{2\pi ny}-e^{2\pi n(1-y)}}{e^{2\pi n}-1}+\cos 2\pi ny\right)dy,$$

$$\varphi_{2,n}=\left(\frac{1}{2\pi}\right)^5\frac{\varphi^{(V)}_{2,n}}{n^5},\quad \varphi^{(V)}_{2,n}=2\int_0^1\frac{\partial^5\varphi(y)}{\partial y^5}\sin 2\pi ny\,dy,$$

$$\psi_{1,n}=-\left(\frac{1}{2\pi}\right)^5\frac{\psi^{(V)}_{1,n}}{n^5},\quad \psi^{(V)}_{1,n}=\int_0^1\frac{\partial^5\psi(y)}{\partial y^5}\left(\frac{e^{2\pi ny}-e^{2\pi n(1-y)}}{e^{2\pi n}-1}+\cos 2\pi ny\right)dy,$$

$$\psi_{2,n}=\left(\frac{1}{2\pi}\right)^5\frac{\psi^{(V)}_{2,n}}{n^5},\quad \psi^{(V)}_{2,n}=2\int_0^1\frac{\partial^5\psi(y)}{\partial y^5}\sin 2\pi ny\,dy,$$

$$f_{1,n}(t)=-\frac{1}{2\pi}\frac{f'_{1,n}(t)}{n},\quad f'_{1,n}(t)=\int_0^1\frac{\partial f(y)}{\partial y}\left(\frac{e^{2\pi ny}-e^{2\pi n(1-y)}}{e^{2\pi n}-1}+\cos 2\pi ny\right)dy,$$

$$f_{2,n}(t)=\frac{1}{2\pi}\frac{f'_{2,n}}{n},\quad f'_{2,n}(t)=2\int_0^1\frac{\partial f(y)}{\partial y}\sin 2\pi ny\,dy.$$

It is easy to see that [16]

$$\left\|\vec{\varphi}^{(V)}_{\kappa}(x)\right\|_{\ell_2}\le C_1\left\|\frac{\partial^5\varphi(x)}{\partial x^5}\right\|_{L_2[0,1]},\quad \left\|\vec{\psi}^{(V)}_{\kappa}(x)\right\|_{\ell_2}\le C_1\left\|\frac{\partial^5\psi(x)}{\partial x^5}\right\|_{L_2[0,1]},\quad \left\|\vec{f}'_{\kappa}(t)\right\|_{\ell_2}\le C_2\left\|\frac{\partial f(t,x)}{\partial x}\right\|_{L_2[0,1]}.$$

Theorem 4. *If there exists a solution of Problem* (1)-(4)*, then the solution is unique.*

Proof. We suppose that there exist two different solutions $\bar{U}_1(t,x)$ and $\bar{U}_2(t,x)$ to the problem (1)-(4). Then the difference $U(t,x)=\bar{U}_1(t,x)-\bar{U}_2(t,x)$ is a solution of the equation (1), satisfying the conditions (2)-(4) with functions $\varphi(x)\equiv 0,\ \psi(x)\equiv 0,\ f(t,x)\equiv 0$. Then, it follows from formulas (19)-(22) and (34) that

$$u_{\kappa,n}(t)=\int_0^1 U(t,y)\omega_{\kappa,n}(y)dy\equiv 0,\quad \kappa=1,2.$$

From this, due to the completeness of the system (13) in the space $L_2[0,1]$, it follows that $U(x,t)=0$ almost everywhere on $[0,1]$ for any $t\in[0,T]$. Since $U\in C(\bar{\Omega})$, it follows that $U(t,x)\equiv 0$ in $\bar{\Omega}$. The Theorem 4 is proved.

Theorem 5. *Let the smoothness conditions S be satisfied. Then the function* (34) *will be belonged to the class of functions* (5).

The proof is similar to the corresponding proof from the works [17, 18].

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