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THE INITIAL VALUE PROBLEM FOR THE HIROTA EQUATION WITH FINITE DENSITY AND SELF-CONSISTENT SERIES SOURCE

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Abstract. In this paper, we consider the problem of integration of the Hirota equation with finite density and self-consistent series source using the inverse spectral problem method for the periodic Dirac operator in the class of periodic infinite-gap functions. In addition, we derive the evolution of spectral data of the periodic Dirac operator and prove that the solution of the initial value problem for the infinite Dubrovin system of differential equations has a unique solution.

Keywords. Hirota equation with finite density and self-consistent source, inverse spectral problem method, periodic Dirac operator, infinite Dubrovin system of differential equations, spectral data, traces formulas.

Chekli zichlikka va qator ko‘rinishdagi moslangan manbaga ega Hirota tenglamasi uchun boshlang‘ich qiymat masalasi

Annotatsiya. Ushbu maqolada chekli zichlikka va qator ko‘rinishdagi moslangan manbaga ega Hirota tenglamasini davriy cheksiz zonali funksiyalar sinfida davriy Dirak operatoriga qo‘yilgan teskari spektral masalalar usuli yordamida integrallash masalasi qaraladi. Shuningdek, davriy Dirak operatori spektral berilganlarining evolyutsiyasini keltirib chiqariladi va cheksiz Dubrovin differensial tenglamalar sistemasiga qo‘yilgan boshlang‘ich qiymat masalasi yagona yechimga ega ekanligi isbotlanadi.

Kalit so‘zlar. chekli zichlikka va qator ko‘rinishdagi moslangan manbaga ega Hirota tenglamasi, davriy Dirak operatori, cheksiz Dubrovin differensial tenglamalar sistemasi, spektral berilganlar, izlar formulalari.

Начальная задача для уравнения Хироты с конечной плотностью и самосогласованным источником в виде ряда

Аннотация. В данной работе рассматривается задача интегрирования уравнения Хироты с конечной плотностью и самосогласованным источником в виде ряда с использованием метода обратной спектральной задачи для периодического оператора Дирака в классе периодических бесконечнозонных функций. Кроме того, выводится эволюция спектральных данных периодического оператора Дирака и доказывается, что решение начальной задачи для бесконечной системы дифференциальных уравнений Дубровина имеет единственное решение.

Ключевые слова: уравнение Хироты с конечной плотностью и самосогласованным источником, метод обратной спектральной задачи, периодический оператор Дирака, бесконечная система дифференциальных уравнений Дубровина, спектральные данные, формулы следов.

I. Introduction and formulation of the problem. In many scientific disciplines, including biology, plasma physics, fluid mechanics, chemical physics, optical fibers and others, nonlinear evolutionary equations are frequently employed as models to explain physical phenomena. In addition, integrating nonlinear partial differential equations using the method of inverse spectral problem is one of the greatest scientific discoveries of the last century and has many practical applications. The first breakthrough in this direction was made by Gardner, Green, Kruskal and Miura in 1976 [1]. They found a general solution to the Cauchy problem for the famous Korteweg-de Vries (KdV) equation using the inverse spectral problem method of scattering theory for the Sturm-Liouville operator. Faddeev [2], Marchenko [3], Levitan [4] and others also studied scattering theory for the Sturm-Liouville operator on the entire axis.

After that, many scientists integrated important nonlinear equations using the method of inverse spectral problems. In particular, Zakharov and Shabat [5] proved that the nonlinear Schrodinger equation (NLS)

$$iu_t \pm 2|u|^2u + u_{xx} = 0,$$

can also be integrated using the inverse scattering problem method(ISPM) in the class of rapidly decreasing functions. Soon, Wadati [6] described an algorithm for finding a solution to the Cauchy problem for the modified KdV(mKdV) equation:

$$u_t \pm 6u^2u_x + u_{xxx} = 0.$$

Authors of the papers [8] and [9] proved that the ISPM can also be applied to the solution of the sine-Gordon (sG) equation:

$$u_{xt} = sinu.$$

In 1973, Hirota [7] obtained the exact $N-$envelope-soliton solutions of the following equation, which is a combination of the complex mKdV and NLS equations and was later called the Hirota equation:

$$i\frac{\partial\psi}{\partial t} + i3\alpha|\psi|^2\frac{\partial\psi}{\partial x} + \beta\frac{\partial^2\psi}{\partial x^2} + i\gamma\frac{\partial^3\psi}{\partial x^3} + \delta|\psi|^2\psi = 0,$$

where $\alpha\beta = \gamma\delta, \alpha, \beta, \gamma, \delta \in R_{+,x\in R,t>0.}$ The Hirota equation explains the behavior of nonlinear waves in some physical systems and is also used in the study of fiber optic systems and in the mechanics of telecommunications. In papers [8]-[9], the following convenient form of the Hirota equation is studied from various perspectives:

$$iu_t + \alpha(u_{xx} \pm 2|u|^2u) + i\beta(u_{xxx} \pm 6|u|^2u_x) = 0, \alpha, \beta \in R, x \in R, t > 0.$$

The inverse scattering problem for the Dirac operator on the whole axis was studied in [10]–[11] and other works. It is well known that the operator L is not self-adjoint; in the "rapidly decreasing" case, it can have spectral singularities that lie in the continuous spectrum and have a finite number of multiple complex eigenvalues. Together with the features of the continuous spectrum, the scattering data of the non-self-adjoint Dirac operator also contain spectral singularities and a discrete spectrum. Considering the above, in papers [12]-[13], integrate the important nonlinear evolutionary equations(KdV, sG) with self-consistent source or load terms, in cases where the eigenvalues of the Dirac operator are simple or multiple.

It is known that finding exact solutions of nonlinear evolution equations in the class of periodic functions directly depends on the number of nontrivial gaps of the operators. For this reason, it is convenient to split the class of periodic functions into two sets:

1) the class of periodic finite-gap functions;
2) the class of periodic infinite-gap functions.

It is known from [14] that if $q(x) = 2acos2x, a \neq 0$, then in the spectrum of the Sturm-Liouville operator $Ly \equiv -y'' + q(x)y, x \in R$ all gaps are open, that is, $q(x)$ is a periodic infinite-gap potential. Similar examples exist for the periodic Dirac operator in [15].

The inverse problem for the Sturm-Liouville operator on the half-line in the class of periodic infinite-gap functions was studied in [16]. In [17]-[18], the solvability of the defocusing Hirota equation with additional terms or finite density was proved in the class of π periodic infinite-gap functions. Using the ideas of above works some nonlinear evolutionary equations such as KdV, NLS, mKdV, mKdV–sG, mKdV equations were integrated in [19]-[28], using the inverse problem for the periodic Sturm-Liouville and periodic Dirac operator in the class of periodic infinite-gap functions.

In 1990, Melnikov [29] integrated the NLS equation with self-consistent source using the inverse scattering problem method. After that, many scientists began to study this area due to the applications of the self-consistent source to nonlinear physics and chemistry.

In this paper, the following nonlinear Hirota type equation with finite density and self-consistent series source

is considered with the following initial and smoothness conditions, respectively

$$\begin{cases} p(x,t)|_{t=0} = p_0(x) \in C^6(R), p_0(x) = p_0(x+\pi), \\ q(x,t)|_{t=0} = q_0(x) \in C^6(R), q_0(x) = q_0(x+\pi), \end{cases} \quad (2)$$

$$\begin{cases} p(x+\pi,t) = p(x,t), q(x+\pi,t) = q(x,t), \\ p(x,t), q(x,t) \in C_{x,t}^{2,1}(t>0) \cap C(t \geq 0), x \in R, t > 0. \end{cases} \quad (3)$$

Here $\delta_n(t)$ is given real sequence of continuous functions that has asymptotes

$$\delta_n(t) = O\left(\frac{1}{1+n^2}\right), n \to \pm\infty. \quad (4)$$

In addition, the functions $\psi^{\pm}(x,\mu,t) = \left(\psi_1^{\pm}(x,\mu,t), \psi_2^{\pm}(x,\mu,t)\right)^T$ are Floquet solutions of the following Dirac system of equations

$$L(t)y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x,t)q(x,t) \\ q(x,t) - p(x,t) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mu\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, x \in R, t > 0, \quad (5)$$

and defined as follows:

$$\psi(x,\mu,t) = c(x,\mu,t) + \frac{s_2(\pi,\mu,t) - c_1(\pi,\mu,t) \mp \sqrt{\Delta^2(\mu) - 4}}{2s_1(\pi,\mu,t)} \cdot s(x,\mu,t), \quad (6)$$

where $c(x,\mu,t) = \left(c_1(x,\mu,t), c_2(x,\mu,t)\right)^T$ and $s(x,\mu,t) = \left(s_1(x,\mu,t), s_2(x,\mu,t)\right)^T$ are solutions of equation (5) with initial conditions $c(0,\mu,t) = (1,0)^T$ and $s(0,\mu,t) = (0,1)^T$ respectively, the function $\Delta(\mu,t) = \Delta(\mu) = c_1(\pi,\mu,t) + s_2(\pi,\mu,t)$ is called the Lyapunov function for system of equations (5).

Using the expression for Floquet solutions (6), it is easy to obtain the following identities:

where $c(x,\tau,\mu,t)$ and $s(x,\tau,\mu,t)$ vector functions are solutions of the Dirac system of differential equations of the form

$$L(\tau,t)y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x+\tau,t)q(x+\tau,t) \\ q(x+\tau,t) - p(x+\tau,t) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mu\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, x, \tau \in R, t > 0,$$

and satisfy the initial conditions $c(0,\tau,\mu,t) = (1,0)^T$ and $s(0,\tau,\mu,t) = (0,1)^T$.

In addition, the following asymptotes formulas hold:

$$s_2(\pi,\tau,\mu,t) - c_1(\pi,\tau,\mu,t) = O\left(\frac{1}{1+\mu}\right), \mu \to \pm\infty,$$

$$s_1(\pi,\tau,\mu,t) + c_2(\pi,\tau,\mu,t) = O\left(\frac{1}{1+\mu}\right), \mu \to \pm\infty.$$

From identities (7) it follows that the right-hand sides of the equations in the system (1) are π periodic functions with respect to x. Using equalities (7), asymptotes (4) and the above relations, we can prove the uniform convergence of the series in the right-hand sides of the equations in the system (1).

In this paper, the nonlinear evolutionary partial differential equation (1) is integrated using two first-order ordinary differential equations. In other words, the algorithm for finding the solution of the Cauchy problem (1)-(3) is determined using the inverse spectral problem method for the following periodic Dirac operator:

$$L(\tau,t)y = \begin{pmatrix} p(x+\tau,t)\dfrac{d}{dx} + q(x+\tau,t) \\ \dfrac{-d}{dx} + q(x+\tau,t) - p(x+\tau,t) \end{pmatrix} y = \mu y, x, \tau \in R, t > 0, \quad (8)$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Let λ_n be the eigenvalues of the periodic or antiperiodic $\left(y(0) = \pm y(\pi)\right)$ problem for system of equations (8). They coincide with the roots of the equation $\Delta(\mu) \mp 2 = 0$. The intervals

$(\lambda_{2n-1}, \lambda_{2n}), n \in Z$, are called gaps. They eigenvalues of the Dirichlet problem for the system of equations (8) with the boundary conditions $y_1(0,\mu,\tau,t) = 0, y_1(\pi,\mu,\tau,t) = 0$ denoted by $\xi_n(\tau,t), n \in Z$, also we have $\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}], n \in Z$.

2. Main results.

2.1. Main theorem.

The theorem expressing the main results of the paper is as follows.

Theorem. If periodic infinite-gap functions $p_0(x), q_0(x)$ satisfy the conditions

$$p_0(x+\pi) = p_0(x) \in C^6(R), q_0(x+\pi) = q_0(x) \in C^6(R),$$

then there exist uniquely determined solutions $p(x,t), q(x,t), x \in R, t > 0$ of the Cauchy problem (1)–(3), which are determined, by the sum of following functional series, respectively:

$$p(x,t) = \sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(x,t) \right), (9)$$

$$q(x,t) = \sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k(\tau,t) h_k\big(\xi(x,t)\big), (10)$$

To prove this theorem, we prove the auxiliary lemmas.

2.2. Dubrovin system of differential equations.

Now we derive an infinite Dubrovin system of differential equations that represent the evolution of the spectral data of the operator $L(\tau,t)$.

Lemma 1. Let $p(x,t), q(x,t), x \in R, t > 0$ be solutions of the Cauchy problem (1)-(3). Then the boundaries of the spectrum $\lambda_n(\tau,t), n \in Z$, of the operator $L(\tau,t)$ do not depend on τ and t, that is, $\lambda_n(\tau,t) = \lambda_n, n \in Z$, and the spectral parameters $\xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1, n \in Z$, satisfy the first and second Dubrovin system of differential equations, respectively:

$$1. \frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1} \sigma_n(\tau,t) h_n\big(\xi(\tau,t)\big)\{p(\tau,t) + \xi_n(\tau,t)\}, n \in Z; (11)$$

$$2. \frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^{n} \sigma_n(\tau,t) h_n\big(\xi(\tau,t)\big) g_n\big(\xi(\tau,t)\big), n \in Z. (12)$$

In addition, the following initial conditions are satisfied

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau), \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \in Z (13)$$

where $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in Z$ are the spectral parameters of the Dirac operator $L(\tau,0)$. The sequences $h_n(\xi)$ and $g_n(\xi), n \in Z$, are determined by the formulas:

$$h_n(\xi) = \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \cdot f_n(\xi),$$

$$f_n(\xi) = \sqrt{\prod_{k=-\infty, k\neq n}^{+\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}},$$

$$g_n(\xi) = [4\xi_n^3 + 4p\xi_n^2 + 2\xi_n(p^2 + q^2 + q_\tau) - p_{\tau\tau} + 2(pq_\tau - p_\tau q) + 2p(p^2 + q^2) - 6\rho^2(p + \xi_n)] +$$

$$+[2\xi_n^2 + 2\xi_n p + p^2 + q^2 + q_\tau - \rho^2] + \sum_{k=-\infty}^{+\infty} \frac{\delta_k(t) s_1(\pi, \mu_k, \tau, t)}{\xi_n - \mu_k}, n \in Z.$$

where $p = p(\tau,t), q = q(\tau,t), \xi \equiv \xi(\tau,t) = (\dots, \xi_{-1}(\tau,t), \xi_0(\tau,t), \xi_1(\tau,t), \dots)$, $\sigma \equiv \sigma(\tau,t) = (\dots, \sigma_{-1}(\tau,t), \sigma_0(\tau,t), \sigma_1(\tau,t), \dots)$ and

$$s_1(\pi,\mu,\tau,t) = \pi \prod_{i=-\infty}^{+\infty} \frac{\xi_i(\tau,t) - \mu}{a_i}, a_i = \begin{cases} 1, i = 0 \\ i, i \neq 0 \end{cases}. (14)$$

2.3. Existence and uniqueness of the solution of the Cauchy problem for the Dubrovin system of differential equations.

To prove the existence and uniqueness of the solution of the Cauchy problem for the Dubrovin system of differential equations (12), (13), we use the following change of variables:

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1}) sin^2 x_n(\tau,t), n \in Z. (15)$$

Using the change of variables (15) and traces formulas (9)-(10) and

$$q^2(\tau,t)+q_\tau(\tau,t)=\sum_{k=-\infty}^{+\infty}\left(\frac{\lambda_{2k-1}^2+\lambda_{2k}^2}{2}-\xi_k^2(\tau,t)\right),$$

we write the Cauchy problem for an infinite Dubrovin system of equations (12), (13) in the following form:

$$\begin{cases}\dfrac{\partial x_n(\tau,t)}{\partial t}=H_n(\dots x_{-1}(\tau,t),x_0(\tau,t),x_1(\tau,t),\dots)\\ x_n(\tau,t)|_{t=0}=x_n^0(\tau)=arcsin\sqrt{\dfrac{\xi_n^0(\tau)-\lambda_{2n-1}}{\lambda_{2n}-\lambda_{2n-1}}},n\in Z.\end{cases}\quad(16)$$

Here

$$H_n\big(x(\tau,t)\big)=(-1)^n\sigma_n^0(\tau)f_n(\dots,\lambda_{-1}+(\lambda_0-\lambda_{-1})sin^2x_0(\tau,t),\dots)\times$$
$$\times g_n(\dots,\lambda_{-1}+(\lambda_0-\lambda_{-1})sin^2x_0(\tau,t),\dots).$$

To study the Cauchy problem (16), we introduce the following Banach space:

$$K=\left\{x=(\dots,x_{-1}(\tau,t),x_0(\tau,t),x_1(\tau,t),\dots):\|x\|=\sum_{n=-\infty}^{+\infty}(1+|n|^2)(\lambda_{2n}-\lambda_{2n-1})|x_n(\tau,t)|<\infty\right\}.$$

Then the Cauchy problems (16) can be written as a single equation in a Banach space K:

$$\begin{cases}\dfrac{\partial x(\tau,t)}{\partial t}=H\big(x(\tau,t)\big)\\ x(\tau,t)|_{t=0}=x^0(\tau)\in K,\end{cases}\quad(17)$$

where $H\big(x(\tau,t)\big)=\big(\dots,H_{-1}\big(x(\tau,t)\big),H_0\big(x(\tau,t)\big),H_1\big(x(\tau,t)\big),\dots\big)$.

It is known from [30] that if the initial functions satisfy the conditions $p_0(x+\pi)=p_0(x)\in C^6(R),q_0(x+\pi)=q_0(x)\in C^6(R)$, then the following asymptotes hold for the eigenvalues of the periodic and antiperiodic problem for the Dirac system of equations and length of the gaps:

Using the asymptotes (18) and the relation

$$\inf_{k\neq n}|\xi_n(\tau,t)-\xi_k(\tau,t)|\geq a>0,$$

we can prove the following lemma.

Lemma 2. The following estimates hold:

$$C_1\leq|\overline{f_n}(x)|\leq C_2,\left|\frac{\partial\overline{f_n}(x)}{\partial x_m}\right|\leq C_3\gamma_m,\quad(19)$$

$$|\overline{g}_n(x)|\leq C_4(|n|^3+1),\left|\frac{\partial\overline{g}_n(x)}{\partial x_m}\right|\leq C_5\gamma_m(|n|^2+1)(|m|^2+1).\quad(20)$$

Here the functions $\overline{f_n}(x)$ and $\overline{g}_n(x)$ are functions formed by replacing the variables (15) with the functions $f_n(\xi)$ and $g_n(\xi)$, respectively, $C_j>0,j=1,2,3,4,5$, do not depend on the parameter m and n, and $x=(\dots,x_{-1},x_0,x_1,\dots)$.

Lemma 3. If the initial functions $p_0(x)$and $q_0(x)$ satisfy the following conditions

$$p_0(x+\pi)=p_0(x)\in C^6(R),q_0(x+\pi)=q_0(x)\in C^6(R),$$

then the vector function $H(x)$ satisfies the Lipschitz condition in the Banach space $\mathbb{K}$, that is, there exists a constant $L=const>0$ such that for arbitrary elements $x,y\in K$ the following inequality holds:

$$\vee H(x)-H(y)\vee\leq L\vee x-y\vee,$$

Where

$$L=A\sum_{n=-\infty}^{+\infty}\gamma_n|n|^3(1+|n|^2)<\infty.$$

2.4. Proof of the Main theorem. According to Lemma 3, there is a unique global solution of the Cauchy problem for the infinite Dubrovin system of differential equations, that is, there are uniquely determined spectral data $\{\lambda_n, \xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1, n \in Z\}$of the operator $L(\tau,t)$.

It is known that the coefficients $p(\tau,t)$ and $q(\tau,t)$ of the operator $L(\tau,t)$, that is, the solutions of the Cauchy problem (1)-(3), are determined uniquely from the spectral data of the operator $L(\tau,t)$. Thus, a global solution of the Cauchy problem (1)-(3) is exists and unique.

2.5. Algorithm for finding a solution of the Cauchy problem for a Hirota-type equation with a self-consistent series source.

Theorem 1 and Lemma 2 provide a method for finding a solution of the initial value problem (1)-(3):

1) First, we find the spectral data $\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in Z$, of the Dirac operator $L(\tau,0)$;

2) Now, solving the Cauchy problem (12), (13) for an arbitrary value of τ, we find $\lambda_n, \xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1, n \in Z$;

3) From the traces formulas (9), (10) we determine the functions $p(\tau,t)$ and $q(\tau,t)$, that is, the solutions to problem (1)-(3);

4) Then, using the found functions $p(\tau,t)$ and $q(\tau,t)$, we find the Floquet solutions $\psi^{\pm}(x,\mu,t) = \left(\psi_1^{\pm}(x,\mu,t), \psi_2^{\pm}(x,\mu,t)\right)^T$ of the system of equations (5).

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