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PLASMA MAGNETOSPHERE OF SLOWLY ROTATING MAGNETIZED NEUTRON STARS IN MODIFIED GRAVITY

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Abstract. This study explores the electromagnetic properties of slowly rotating magnetized neutron stars within Modified Gravity (MOG), focusing on their plasma magnetospheres. Building on vacuum solutions of Maxwell's equations by Deutsch and Rezzolla, it incorporates plasma effects as per Goldreich and Julian, analyzing charge-separated plasma and electric field amplification. Analytical solutions for the Goldreich-Julian charge density, adjusted for MOG parameters, reveal enhanced magnetic field components and potential pulsar jet formation. The influence of the MOG parameter α on polar cap geometry is investigated, bridging theoretical models with astrophysical observations. These findings offer new insights into the interplay of gravitational theories and pulsar electrodynamics.

Keywords: magnetosphere, neutron star, plasma.

O‘zgartirilgan gravitatsiyada sekin aylanishli magnitlangan neytron yulduzlarning plazma magnitosferasi

Annotatsiya. Ushbu tadqiqot o‘zgartirilgan gravitatsiya (MOG) doirasida sekin aylangan magnitlangan neytron yulduzlarining elektromagnit xususiyatlarini o‘rganadi, e’tiborni plazma magnitosferalariga qaratadi. Deutsch va Rezzolla tomonidan ishlab chiqilgan Maxwell tenglamalarining vakuum yechimlari Goldreich va Julian tomonidan keltirilgan plazma effektlari bilan kengaytirilib, zaryad ajratilgan plazma va kuchaygan elektr maydonlari tahlil qilinadi. MOG parametrlariga moslashtirilgan Goldreich-Julian zaryad zichligi uchun analitik hisoblar magnit maydon komponentlarining kuchayishini va pulsar jetlarining kelib chiqishini ko‘rsatadi. MOG parametri α ning polar qalpoq tuzilmasiga ta’siri o‘rganilib, nazariy modellar astrofizik kuzatuvlar bilan bog‘lanadi. Natijalar gravitatsiya nazariyalari va pulsar elektrodinamikasi o‘zaro ta’siriga yangi nuqtai nazarni taklif qiladi.

Kalit so‘zlar: magnitosfera, neytron yulduz, plazma.

Плазменная магнитосфера медленно вращающихся магнитных нейтронных звезд в модифицированной гравитации

Аннотация. Данное исследование изучает электромагнитные свойства медленно вращающихся магнитных нейтронных звёзд в рамках модифицированной гравитации (MOG), уделяя внимание их плазменным магнитосферам. Оно расширяет вакуумные решения уравнений Максвелла, разработанные Дойчем и Реццоллой, интегрируя эффекты плазмы, описанные Голдрейхом и Джулианом, с акцентом на разделённую зарядами плазму и усиленные электрические поля. Аналитические вычисления плотности заряда Голдрейха-Джулиана, адаптированные к параметрам MOG, указывают на усиление компонентов магнитного поля и

возможное происхождение джетов пульсаров. Влияние параметра MOG α на структуру полярной шапки анализируется, связывая теоретические модели с астрономическими данными. Результаты предлагают новые взгляды на взаимодействие гравитационных теорий и электродинамики пульсаров.

Ключевые слова: магнитосфера, нейтронная звезда, плазма.

Introduction. The study of electromagnetic fields in the vicinity of slowly rotating neutron stars provides critical insights into their complex behavior under general relativistic conditions. By solving Maxwell's equations in a vacuum environment external to the star, analytical models delineate the electric and magnetic field structures, assuming a simplified magnetosphere devoid of plasma. These solutions serve as foundational benchmarks for more advanced theoretical and numerical frameworks. Early work by Deutsch [1] derived analytical expressions for the electromagnetic fields of an inclined rotator in flat spacetime, laying the groundwork for subsequent investigations. These results were later refined to account for spacetime curvature, improving their relevance to astrophysical systems [2].

Building on this foundation, Rezzolla and colleagues [2] advanced the methodology by addressing Maxwell's equations in the curved spacetime of a slowly rotating neutron star with a misaligned magnetic dipole. Their work incorporates frame-dragging effects, which significantly modify the electromagnetic field configurations near the stellar surface and induce an electric field essential for pulsar electrodynamics. In parallel, Muslimov and Tsygan [3] extended these models to include a plasma-filled magnetosphere, revealing how spacetime curvature amplifies electric fields in the polar cap regions, thereby influencing charged particle acceleration.

Recent developments by Rezzolla and Ahmedov [4] have further generalized these solutions to encompass multipolar field components in oscillating magnetized neutron stars. Their findings highlight the role of higher-order fields in energy dissipation through the Poynting flux, offering valuable perspectives on pulsar radiative processes. Despite their utility, vacuum solutions rely on the idealized absence of plasma, necessitating further exploration of realistic magnetospheric environments to align theoretical predictions with observational data.

The introduction of a plasma-filled magnetosphere fundamentally alters the vacuum electromagnetic framework, fostering a dynamic interaction between relativistic plasma and strong magnetic fields. In their pioneering 1969 model, Goldreich and Julian [5] posited that a rotating, magnetized neutron star generates a charge-separated plasma that co-rotates with the star up to the light cylinder. This plasma mitigates the longitudinal electric field, reducing its strength relative to vacuum conditions, a phenomenon thoroughly examined in curved spacetime by Muslimov and Harding [6]. Beskin [7] underscored the impact of frame-dragging on the Goldreich–Julian charge density, enhancing particle acceleration within the plasma environment.

Moreover, analytical solutions by Rayimbaev et al. [8] within Einstein-aether gravity refine this charge density, demonstrating how alternative gravitational theories shape plasma dynamics. Pétri's research on pulsar electrospheres [9] further reveals that differential rotation in plasma disks triggers instabilities, such as the diocotron mode, which govern the magnetosphere's long-term evolution. Collectively, these studies emphasize the magnetosphere's pivotal role in powering pulsar emissions, with plasma feedback self-consistently shaping electromagnetic fields beyond vacuum approximations.

This work aims to extend these frameworks by exploring electromagnetic field solutions in the context of alternative gravitational theories, offering new analytical insights into the interplay between spacetime curvature, plasma dynamics, and pulsar electrodynamics. Our findings seek to bridge the gap between theoretical models and astrophysical observations, contributing to a deeper understanding of neutron star magnetospheres.

In this study, we adopt the spacetime metric signature (−, +, +, +) and employ a geometrized unit system where the gravitational constant and the speed of light are set to unity (G_N=c=1). For astrophysical applications, however, the speed of light is explicitly included in our derived expressions to ensure clarity and applicability. Latin indices are used to denote spatial components, ranging from 1 to 3, while Greek indices encompass both temporal and spatial components, ranging from 0 to 3.

Vacuum solutions of maxwell equations. In this section, we study the vacuum electromagnetic field structure of slowly rotating magnetized neutron stars in

A. Metric spacetime exterior to neutron stars

The spacetime around slowly rotating magnetized neutron stars in Modified Gravity (MOG) can be expressed in spherical coordinates (*t, r, θ, φ*) by the following metric,

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 - 2\omega r^2sin^2\theta dtd\varphi (1)$$

with the lapse function

$$N^2 = 1 - \frac{2M}{r} + \frac{\alpha M}{r^2} (2)$$

where M is the mass of the NS with radius R. The frame-dragging angular velocity in the MOG can be expressed in the following form.

$$\omega = \frac{2J}{r^3} (3)$$

The total angular momentum of a neutron star, denoted as $J=I\Omega$, is determined by its moment of inertia I and angular velocity Ω. Within the framework of the MOG model, the angular momentum of the star, J_{MOG}, exhibits negligible deviation from that predicted by general relativity, J_{GR}, such that $J_{MOG}=J_{GR}=J$, which remains constant [16]. Observational measurements of the moment of inertia in neutron stars pose significant challenges for relativistic astrophysics due to the complexity of direct empirical assessments. Nevertheless, in theoretical analyses, the moment of inertia can be approximated as $I=\beta MR^2$, where β represents a normalization coefficient, M is the stellar mass, and R is the radius. In the Newtonian limit for a uniform sphere, the coefficient takes the value $\beta=2/5$.

B. Vacuum solutions of the Maxwell equations.

We now proceed to formulate the general relativistic versions of Maxwell's equations within spacetime, as outlined in Equation (1). The Maxwell equations for the electromagnetic field tensor $F^{\mu\nu}$ and its dual counterpart $*F^{\mu\nu}$ are expressed in tensor form as follows:

$$\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 4\pi\sqrt{-g}J^\nu, \partial_\mu\left(\sqrt{-g} * F^{\mu\nu}\right) = 0 (4)$$

Where g represents the determinant of the metric, and the magnetic moment of the star is considered static (time-independent), with the medium inside the star exhibiting superconducting properties (conductivity $\sigma\to\infty$). Additionally, the rotational effects of the star are sufficiently minor to neglect their impact on the overall formulation.

Following the approach outlined in Ref. [2], we seek solutions to the Maxwell equations for the magnetic field components by employing a method of separating variables. The magnetic field components are defined in the following form:

$$B^{\hat{r}}(r,\theta,\varphi) = F(r)(cos\chi cos\theta + sin\chi sin\theta cos\lambda) (5)$$

$$B^{\hat{\theta}}(r,\theta,\varphi) = G(r)(cos\chi sin\theta - sin\chi cos\theta cos\lambda) (6)$$

$$B^{\hat{\phi}}(r,\theta,\varphi) = H(r)sin\chi sin\lambda (7)$$

where $\lambda(t)=\varphi-\Omega t$ denotes the azimuthal angle adjusted for rotation, and the functions F(r), G(r), and H(r) (hereafter referred to as F, G, and H) represent the radial components of the magnetic field. These components account for MOG corrections and incorporate rotation effects. Here, χ signifies the angle between the magnetic dipole axis and the rotation axis, commonly termed the inclination angle. After performing straightforward algebraic manipulations, we have derived a system of differential equations governing the radial functions

$$(r^2F)_{,r} + \frac{2r}{N}G = 0 (8)$$

$$(rNH)_{,r} + F = 0 (9)$$

$$H = G (10)$$

From this stage onward, to streamline our subsequent computations and approximations, we adopt the following substitutions: the mass-to-radius ratio is defined as $\epsilon=M/R$, where R represents the dimensionless radial coordinate. The lapse function, as presented in Equation (7), can be reformulated as:

$$N^2 = 1 - \frac{(\alpha+1)\epsilon}{\eta} + \frac{\alpha(\alpha+1)\epsilon^2}{4\eta^2} (11)$$

where η denotes a scaled radial coordinate. This adjustment facilitates the analysis of the spacetime geometry surrounding neutron stars, consistent with the modified gravity framework discussed earlier.

By applying this formulation, we readily derive a second-order ordinary differential equation for the radial component F, utilizing the system of equations (8)-(10). The resulting equation takes the following form:

$$\frac{d}{d\eta}\left(N^2(\eta,\varepsilon)\frac{d}{d\eta}(r^2F)\right)-2F=0 \quad (12)$$

The specific solution for the radial function F(η) is given by:

$$F(\eta)=\frac{-3\left(2\sqrt{(1+\alpha)\varepsilon^2}(\varepsilon+\alpha\varepsilon+\eta)+(-\alpha(1+\alpha)\varepsilon^2+\eta^2)log\left[\frac{\varepsilon+\alpha\varepsilon+\sqrt{(1+\alpha)\varepsilon^2}-\eta}{\varepsilon+\alpha\varepsilon-\sqrt{(1+\alpha)\varepsilon^2}-\eta}\right]\right)}{8\left((1+\alpha)\varepsilon^2\right)^{3/2}\eta^2} \quad (13)$$

with the auxiliary functions H(η) and G(η) expressed as:

$$G(\eta)=H(\eta)=\frac{3\left(-2\sqrt{(1+\alpha)\varepsilon^2}(\varepsilon+\alpha\varepsilon-\eta)+(\alpha(1+\alpha)\varepsilon^2-2(1+\alpha)\varepsilon\eta+\eta^2)log\left[\frac{\varepsilon+\alpha\varepsilon+\sqrt{(1+\alpha)\varepsilon^2}-\eta}{\varepsilon+\alpha\varepsilon-\sqrt{(1+\alpha)\varepsilon^2}-\eta}\right]\right)}{8\left((1+\alpha)\varepsilon^2\right)^{3/2}\eta^2\sqrt{\frac{\alpha(1+\alpha)\varepsilon^2-2(1+\alpha)\varepsilon\eta+\eta^2}{\eta^2}}} \quad (14)$$

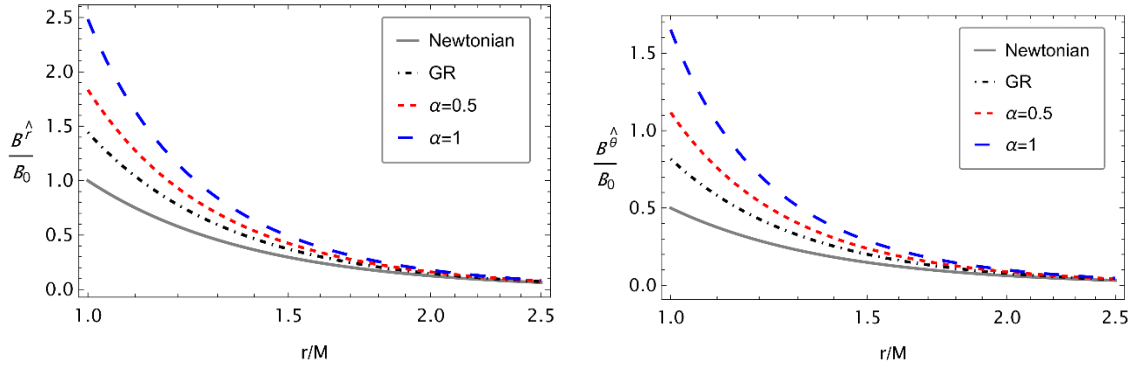

FIGURE1: Radial dependence of orthonormal radial (in the left panel) and angular (in the right panel) components of magnetic field on the polar cap of the neutron star with ε = 0.2 to normalized to its surface value in Newtonian for different values of MOG parameter.

Equation (13) represents a linear, homogeneous, second-order ordinary differential equation, which can be effectively addressed using standard techniques such as series expansion. After conducting detailed computations that integrate the solution of Equation (13) with the system of differential equations (8)-(10), we have derived the necessary framework.

To enhance the interpretability of numerical values and streamline the evaluation of the Goldreich-Julian charge density alongside the Poisson equation for the scalar electric field potential, we redefine the components of the magnetic field associated with a slowly rotating magnetized neutron star. This redefinition employs new dimensionless coordinates as outlined below.

$$B^{\hat{r}}(r,\theta,\varphi)=\frac{B_0}{\eta^3}\frac{f(\eta;\varepsilon)}{f(1;\varepsilon)}(cos\chi cos\theta+sin\chi sin\theta cos\lambda) \quad (15)$$

$$B^{\hat{\theta}}(r,\theta,\varphi)=\frac{B_0}{\eta}\frac{N(\eta;\varepsilon)}{\eta^2}\left[\frac{3}{2f(1;\varepsilon)N^2(\eta;\varepsilon)}-\frac{f(\eta;\varepsilon)}{f(1;\varepsilon)}\right](cos\chi sin\theta-sin\chi cos\theta cos\lambda) \quad (16)$$

$$B^{\hat{\phi}}(r,\theta,\varphi)=\frac{B_0}{\eta}\frac{N(\eta;\varepsilon)}{\eta^2}\left[\frac{3}{2f(1;\varepsilon)N^2(\eta;\varepsilon)}-\frac{f(\eta;\varepsilon)}{f(1;\varepsilon)}\right]sin\chi sin\lambda \quad (17)$$

Where $B_0 = 2\mu/R^3$ is the Newtonian value of the magnetic field at the surface on the pole of the star with dipole moment μ and χ is the inclination angle between the magnetic field and the rotation axis of the neutron star.

The radial function in the expressions (15)-(17) has the form

$$f(\eta;\varepsilon)=\frac{-3}{8}\left(\frac{1+\alpha}{1+2\alpha}\right)^{\frac{3}{2}}\left(\frac{r}{\varepsilon}\right)^{3}\left(\sqrt{\frac{1+2\alpha}{1+\alpha}}\frac{2\varepsilon}{r}\left(1+(1+\alpha)\frac{\varepsilon}{r}\right)\right.$$

$$\left.+\left(1+\frac{\epsilon^{2}\alpha}{(1+\alpha)r^{2}}\right)log\left[\frac{(1+\alpha+\sqrt{1+\alpha})\varepsilon-r}{(1+\alpha-\sqrt{1+\alpha})\varepsilon-r}\right]\right)(18)$$

which coincides with the results in Refs. [3, 6]. To compare behaviour of the magnetic field of the neutron star in MOG and GR, one can compare the profiles of the magnetic field lines.
Figure 1 shows the radial dependence of the magnetic field components on the polar cap of the neutron star normalized to its surface magnetic field calculated in Newtonian. As seen on both panels of the figure, in the presence of MOG parameter the values of both components of the magnetic field increase.

Goldreich-julian charge density in the mog. In this section, we examine the plasma magnetosphere surrounding a slowly rotating neutron star within the framework of MOG. Functioning primarily as a rotating magnetic dipole, the neutron star generates an electric field, which induces electric charges on its surface. These charges are subsequently ejected into the surrounding space, where they undergo acceleration due to the Lorentz force, emitting gamma-ray photons that trigger a cascade of electron-positron pair creation. Under typical conditions, these pairs recombine; however, in the proximity of rotating compact objects, the resulting charges are separated by frame-dragging effects. Consequently, the neutron star becomes enveloped by a dense plasma layer characterized by a distinct distribution of electrons and positrons.

It is hypothesized that this spatial charge distribution generates an additional electric field aligned with the magnetic field lines, potentially driving the acceleration of charges away from the star and contributing to the formation of pulsar jets. This supplementary induced charge is commonly referred to as the Goldreich-Julian charge [5,10-11]. The general expression for the Goldreich-Julian charge density ρ_{GJ} in the context of MOG can be formulated as follows [3].

$$-4\pi\rho_{GJ}=\nabla\left(\frac{1}{N(\eta;\varepsilon)}g\times B\right)(19)$$

where $g_i=-g_{0i}/g_{00}$ can be found from the spacetime metric in [12].
Following [3] we have

$$-4\pi\rho_{GJ}=\nabla\left\{\frac{1}{N(\eta;\varepsilon)}\left[1-\frac{\kappa}{\eta^{3}}\left(1-2(1+\alpha)\frac{\varepsilon}{r}-\frac{\alpha(1+\alpha)\varepsilon^{2}}{r^{2}}\right)\right]u\times B\right\}(20)$$

with $\kappa=\varepsilon\beta$ and $u=\Omega\times r$,
As we are only interested in the polar cap region, wecalculate the Goldreich-Julian charge density using the small polar angle approximation $\xi \ll 1$. Substituting the expressions (15)-(16) into the equation (20) and doing some algebraic transformations one can easily obtain the
expression for the Goldreich-Julian charge density ρ_{GJ} in MOG in the following form

$$\rho_{GJ}=\frac{\Omega B_{0}}{2\pi cN\eta^{3}}\frac{f(\eta)}{f(1)}\left\{\left[1-\frac{\kappa}{\eta^{3}}\left(1+\frac{\varepsilon\alpha}{(1+\alpha)2\eta}\right)\right]cos\chi+\frac{3}{2}H(\eta;\varepsilon)\theta sin\chi cos\varphi\right\}(21)$$

$$H(\eta;\varepsilon)=\frac{1}{K}\left\{\frac{\varepsilon}{\eta}\left(1+\frac{\varepsilon\alpha}{(1+\alpha)2\eta}\right)-\frac{\kappa}{\eta^{3}}\left[1+2\varepsilon\frac{\alpha}{3(1+\alpha)\eta}\left(1+\frac{\varepsilon}{\eta}+\frac{\alpha\varepsilon^{2}}{2(1+\alpha)\eta^{2}}\right)\right]\right.$$

$$\left.+\frac{1}{f(\eta)N^{2}}\left[1+\frac{\kappa}{\eta^{3}}-\frac{\varepsilon}{\eta}\left(3+\frac{\varepsilon\alpha}{(1+\alpha)\eta}\left(1-\frac{1}{5\eta^{2}}\right)\right)\right]\right\}(22)$$

Where: $K=1+\frac{\varepsilon^{2}\alpha}{(1+\alpha)\eta^{2}}$
One can easily demonstrate that the expression (21) for the Goldreich-Julian charge density and the function (23) in α = 0 (GR) case take the form obtained in [3] in GR.

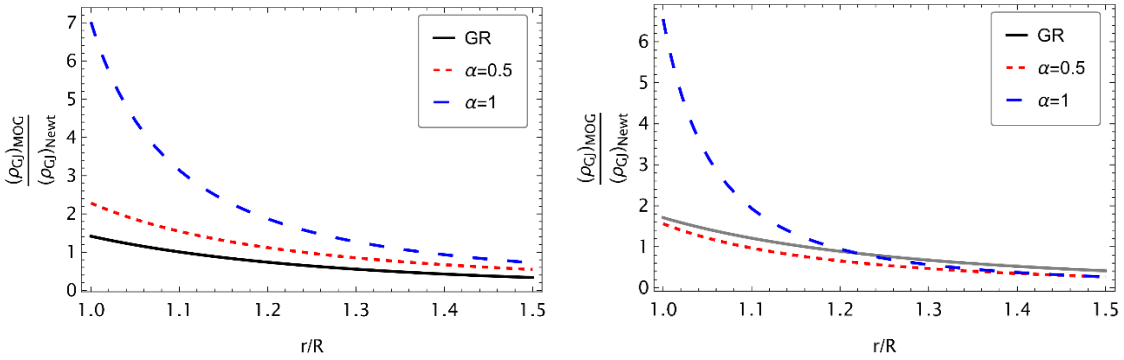

FIGURE 2: Illustrates the radial dependence of the ratio of charge densities in MOG to those in general relativity (GR) for various values of the MOG parameter α \alpha α, with a stellar compactness of ε=0.2. The left panel corresponds to an inclination angle χ=0, while the right panel depicts χ=π/2, highlighting the influence of magnetic alignment on charge distribution.

The magnetic flux emanating from the stellar surface is a critical parameter for understanding the magnetospheric structure. This quantity is derived as [3]:

$$\Psi(\eta;\varepsilon) = \frac{\pi R^2 B_0}{\eta}\frac{f(\eta;\varepsilon)}{f(1;\varepsilon)} (23)$$

where R is the stellar radius, B_0 represents the surface magnetic field strength in the Newtonian limit, and $f(\eta;\varepsilon)$ is the radial function defined in Equation (18). The conservation of magnetic field lines allows us to establish the following relationship:

$$\Psi(\eta;\varepsilon) sin^2\theta = \Psi(1;\varepsilon) sin^2\theta_0 (24)$$

where θ_0 denotes the magnetic colatitude of a field line at the stellar surface. Based on Ref. [13], the evolution of the colatitude θ with radial distance is governed by:

$$\frac{d\theta(\eta)}{d\eta} = \frac{B^{\hat{\theta}}}{B^{\hat{r}}} (25)$$

A useful relation between θ and θ_0 can be obtained in the following form [3]

$$\theta(\eta;\varepsilon) = \xi\Theta(\eta;\varepsilon) = sin^{-1}\left(sin(\xi\Theta)\sqrt{\eta\frac{f(1;\varepsilon)}{f(\eta;\varepsilon)}}\right) (27)$$

$$\Theta(\eta;\varepsilon) = \Theta_0\sqrt{\eta\frac{f(1;\varepsilon)}{f(\eta;\varepsilon)}} (28)$$

For convenience, along with the variable η, one can introduce a new dimensionless variable $\xi = \theta/\Theta$ in the range $(0 \le \xi \le 1)$ that labels the open field lines. Here, Θ is the polar angle (magnetic colatitude) of a radius vector sliding along the last open field line. Θ_0 is the magnetic colatitude of the last open field line on the stellar surface. It can be defined as $\Theta_0 = \sin^{-1}(R/R_{LC})^{1/2}$ where $R_{LC} = c/\Omega$ is the radius of the light-cylinder.

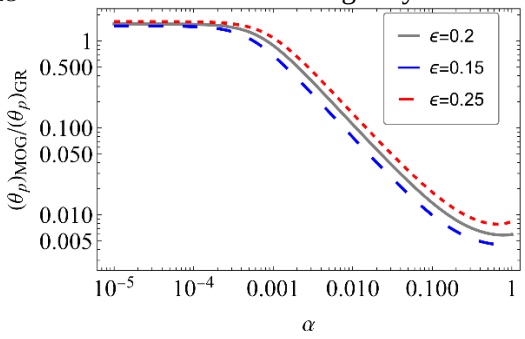

FIGURE 3: Presents the variation of the polar cap size as a function of the MOG parameter for different compactness parameters ε. This analysis reveals how modifications in gravitational theory affect the geometry of the open field line region, providing insights into the magnetospheric structure.

The derived magnetic flux and colatitude relations underscore the role of MOG parameters in shaping the magnetospheric dynamics. The observed increase in charge density ratios and polar cap size with α suggests enhanced electromagnetic activity, which may correlate with observed pulsar jet phenomena. These findings extend the vacuum solutions discussed earlier, integrating plasma effects and offering a pathway for future observational validation.

Conclusion. This investigation into the plasma magnetosphere of slowly rotating magnetized neutron stars within the Modified Gravity (MOG) framework has yielded significant insights into their electromagnetic behavior. By extending vacuum solutions of Maxwell's equations to include plasma effects, we have demonstrated that spacetime curvature and frame-dragging, as modified by MOG parameters, play a crucial role in shaping the magnetospheric structure. The derived expressions for the Goldreich-Julian charge density highlight the amplification of electric fields in the polar cap region, driven by the separation of electron-positron pairs, which may underpin the formation of pulsar jets. Comparative analyses of magnetic field components and charge density ratios between MOG and general relativity underscore the influence of the parameter α \alpha α, with increased values correlating with enhanced electromagnetic activity and altered polar cap geometries.

These findings validate the utility of alternative gravitational theories in refining our understanding of neutron star magnetospheres, aligning theoretical predictions with observed pulsar phenomena. However, the idealized assumptions, such as the neglect of complex plasma interactions and external magnetic fields, suggest avenues for future research. Numerical simulations incorporating realistic magnetospheric conditions and comparisons with observational data from pulsar timing arrays could further elucidate these dynamics. This study lays a robust foundation for advancing the field, bridging theoretical advancements with empirical validations in relativistic astrophysics.

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