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NUMERICAL RANGE OF THE FRIEDRICHS MODEL WITH RANK THREE PERTURBATION

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Abstract. In this paper the Friedrichs model with rank three perturbation, acting in the Hilbert space $L_2[-\pi;\pi]$ is considered. This model is a linear, bounded, and self-adjoint and it is a model operator corresponding to the Hamiltonian of the two-particle system on the one-dimensional lattice. The numerical range of the considered Friedrichs model is studied by the numerical ranges of the three auxiliary Friedrichs models with rank one perturbation.

Keywords: Friedrichs model, perturbation operator, integral operator, lattice, system of particles, eigenvalue, numerical range.

Uch o‘lchamli qo‘zg‘alishga ega Fridrixs modelining sonli tasviri

Annotatsiya. Ushbu maqolada $L_2[-\pi;\pi]$ Hilbert fazosida aniqlangan uch o‘lchamli qo‘zg‘alishga ega Fridrixs modeli qaralgan. Bu model chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma operator bo‘lib, bir o‘lchamli panjaradagi ikkita zarrachali sistema Hamiltonianiga mos model operatordir. Tadqiq qilinayotgan Fridrixs modelining sonli tasviri yordamchi uchta bir o‘lchamli qo‘zg‘alishga ega Fridrixs modellari sonli tasvirlari yordamida o‘rganilgan.

Kalit so‘zlar: Fridrixs modeli, qo‘zg‘alish operatori, integral operator, bir o‘lchamli panjara, zarrachalar sistemasi, xos qiymat, sonli tasvir.

Числовая область значений модели Фридрихса с трехмерным возмущением

Аннотация. В данной статье рассматривается модель Фридрихса с трехмерным возмущением действующий в Гильбертовом пространстве $L_2[-\pi;\pi]$. Этот модель является линейным, ограниченным и самосопряженным оператором и модельным оператором соответствующий гамильтониану системы двух частиц на одномерной решетке.

Ключевые слова: модель Фридрихса, оператор возмущения, интегральный оператор, одномерная решетка, система частиц, собственное значение, числовая область значений.

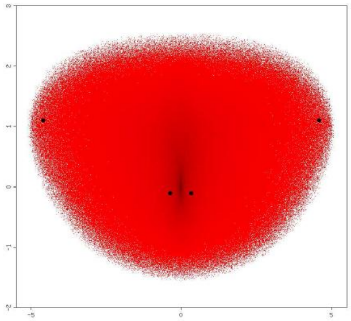
1. Introduction: the numerical range of the linear operator. In the spectral theory of linear operators, the concept of spectrum is important. In many cases, we are faced with the problem of determining not the spectrum of a linear operator, but the domain where it is located. It is well known from the course of functional analysis that the spectrum of a linear operator lies in the complex plane. If the linear operator A defined in the Hilbert space X is bounded, then the spectrum of the operator A is contained in the closed circle centered at the origin and with radius $||A||$. If in addition the operator A is self-adjoint, then its spectrum is contained in the closed segment $[-||A||; ||A||]$. In

order to improve these results, the concept of the numerical range of a linear operator is introduced. For the reader's convenience when reading the text of the paper, we provide some information related to the introduction and study of the concept of numerical range.

Let A be the linear bounded operator defined in the complex Hilbert space X. By sympols $||\cdot||$ and $(\cdot,\cdot)$ we denote the norm and scalar product in X, respectively. The set of the form

$$W(A) := \{(Ax, x): x \in X: ||x|| = 1\}$$

is called the numerical range of the operator A. This concept was introduced for the first time in 1918 by Toeplitz in the paper [1] for matrices, and it was proved that all the eigenvalues of the matrix, that is, its spectrum, lie in the numerical range of the given matrix, and the boundary of the numerical range is a convex line. In his paper [2], Hausdorff showed that a numerical range is convex as a set. To make it more understandable to the reader, we give an example of the numerical range of the matrix from a geometric point of view [3]. In the following picture



the numerical range of the matrix

$$M := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & i & 5i \\ -1 & -2 & -5i & i \end{pmatrix}$$

is given.

Later, in the paper [4] it is shown that the above-mentioned properties are valid not only for matrices, but also for an arbitrary linearly bounded operator, and the spectrum of a linearly bounded operator A defined in the Hilbert space X lies in the closure of its numerical range, that is, the relation $\sigma(A) \subset \overline{W(H)}$ is valid.

We describe another important property of the numerical range of a linearly bounded operator:

$$W(A) \subset \{x \in X: ||x|| \leq ||A||\}.$$

It can be seen that the closure of the numerical range of a linearly bounded operator A is a set "smaller" than the circle $\{x \in X: ||x|| \leq ||A||\}$ containing its spectrum.

Two main issues can be distinguished from the above considerations. First, under what conditions is the numerical range of a linearly bounded operator a closed set? Second, under what conditions do the numerical range and spectrum of a linearly bounded operator coincide? In the paper [5] for the Friedrichs model with two-dimensional perturbation, and in the paper [6] for the generalized Friedrichs model, the conditions under which the numerical range and the spectrum coincide were found. In addition, there are cases where the numerical range is a closed set. The methods of the theory of threshold phenomena were used in the conducted studies.

In the present paper the spectrum and the numerical range of the linear, bounded and self-adjoint Friedrichs model with rank three perturbation in the Hilbert space $L_2[-\pi;\pi]$ is studied. It should be noted that the studied Friedrichs model can also be considered as a Hamiltonian corresponding to a two-particle system on a lattice.

II. The Friedrichs model and its spectrum. The study of the spectrum of the Friedrichs model is one of the topical issues of quantum mechanics, statistical mechanics and hydrodynamics. The spectral properties of the Friedrichs model have been analyzed in many works. We will touch on some of them. In the paper [7], the case where the essential spectrum of the Friedrichs model has a gap is analyzed and the formula for determining the number of eigenvalues located in an arbitrary interval outside its essential spectrum is derived. In the paper [8], the existence conditions for the eigenvalues

of the lattice Friedrichs model with two-dimensional perturbation were found, and this result was applied to the exact calculation of the eigenvalues of the Hamiltonian corresponding to the three-particle system. The paper [9] is devoted to the study of the eigenvalues of the family of Friedrichs models with one-dimensional perturbation. In this case, the asymptotic expansion for the Freddholm determinant associated with the Friedrichs model having a virtual level at the zero point or the zero being an eigenvalue is obtained. In the paper [10] it is considered the case where the dispersion function has a special form and has a constant minimum at several points, and it is found the necessary and sufficient conditions for the Friedrichs model to be a zero-energy resonance or a zero eigenvalue.

Now we introduce the Friedrichs model that will be studied in this paper. In the Hilbert space $L_2[-\pi;\pi]$ we consider the operator of the form

$$H = H_0 - V_1 - V_2 - V_3. \tag{1}$$

Here the non-perturbed operator H_0 is the multiplication operator by the function $u(\cdot)$:

$$(H_0 f)(x) = u(x)f(x), \quad f \in L_2[-\pi;\pi],$$

the perturbation operators V_α are integral operators of the form:

$$(V_\alpha f)(x) = v_\alpha(x)\int_{-\pi}^{\pi} v_\alpha(t)f(t)dt, \quad \alpha = 1,2,3, \qquad f \in L_2[-\pi;\pi].$$

The parameter functions $u(\cdot)$, $v_1(\cdot)$, $v_2(\cdot)$ and $v_3(\cdot)$ are real-valued continuous functions defined on $[-\pi;\pi]$ in addition, the parameter functions $v_1(\cdot)$, $v_2(\cdot)$ and $v_3(\cdot)$ are linearly independent functions.

Under these conditions for the parameter functions the operator H defined in the Hilbert space $L_2[-\pi;\pi]$ by the equality (1) is linear, bounded and self-adjoint.

In the modern mathematical physics, the investigated operator is called the Friedrichs model. The multiplication operator H_0 is called non-perturbed operator, the integral operator $V := V_1 + V_2 + V_3$ is called perturbation operator. Since the parameter functions $v_1(\cdot)$, $v_2(\cdot)$ and $v_3(\cdot)$ are linearly independent, the operator V has a rank three. Therefore, the operator H is called the Friedrichs model with rank three perturbation.

It is known that, the two-particle discrete Schroedinder operator h in the quasi-momentum representation is acting in the Hilbert space $L_2(T^2)$ (T^2 is the two-dimensional torus). Upon isolating the total quasi-momentum of the system k the operator h can be decomposed into the direct integral (see [11,12])

$$h = \int_T \oplus\, h(k)dk,$$

where linear bounded operator $h(k), k \in T$ is acting in the Hilbert space $L_2(G_k)$ ($G_k \subset T$ is some manifold). Note that, the Friedrichs model H enjoys the main spectral properties of the two-particle discrete Scroedinger operator $h(0)$. For this reason, the Friedrichs model H can be considered as a Hamiltonian of the system of two-particles on a one-dimensional lattice.

Using the Weyl theorem on the stability of the essential spectrum under finite rank perturbations [13] we obtain that for the essential spectrum of the Friedrichs model H the equality

$$\sigma_{ess}(H) = \sigma(H_0) = \sigma_{ess}(H_0) = [m;M]$$

holds. Here the numbers m and M are defined by

$$m := \min_{x\in[-\pi;\pi]} u(x), \qquad M := \min_{x\in[-\pi;\pi]} u(x).$$

In order to define the eigenvalues and the numerical range of the Friedrichs model H in the domain $\mathbb{C}\backslash[m,M]$ we introduce the regular function of the form

$$I_{\alpha\beta}(z) = \int_{-\pi}^{\pi} \frac{v_\alpha(t)\, v_\beta(t)}{u(t)-z}dt, \quad \alpha,\beta = 1,2,3.$$

Simple discussions show that the number $z \in \mathbb{C}\backslash[m,M]$ is an eigenvalue of the Friedrichs model H if and only if

$$\Delta(z):=\begin{vmatrix} 1-I_{11}(z) & -I_{12}(z) & -I_{13}(z) \\ -I_{21}(z) & 1-I_{22}(z) & -I_{23}(z) \\ -I_{31}(z) & -I_{32}(z) & 1-I_{33}(z) \end{vmatrix}=0.$$

From here for the discrete spectrum of the Friedrichs model H we obtain the equality $\sigma_{\text{disc}}(H)=\{z\in\mathbb{C}\backslash[m,M]:\Delta(z)=0\}$.

Therefore, for the spectrum of the Friedrichs model H the equality

$\sigma(H)=[m;M]\cup\{z\in\mathbb{C}\backslash[m,M]:\Delta(z)=0\}$

holds.

Since the operator $V_1+V_2+V_3$ is positive definite for any number $z\geq M$ and element $f\epsilon L_2[-\pi;\pi]$ the assertion

$\big((H-z)f,f\big)=\big((H_0-z)f,f\big)-((V_1+V_2+V_3)f,f)<0$

is valid. From the last fact we conclude that the Friedrichs model H has not any eigenvalues bigger than M. Therefore, the equalities

$$\max\sigma(H)=\max\sigma_{\text{ess}}(H)=M$$

hold.

III. Numerical range of the Friedrixs model. In this section we investigate the numerical range of the Friedrichs model H.

We formulate the first result of the paper.

Theorem 1. a) The assertion $M\notin W(H)$ holds.

b) The number $z=M$ is a limit point of the set $W(H)$.

Proof. To prove assertion a) of Theorem 1, we assume the opposite, that is, let $M\in W(H)$. Then there exists an element $f\epsilon L_2[-\pi;\pi]$ such that $||f||=1$ and $(Hf,f)=M$. In this case we have

$$(Hf,f)-M=\int_{-\pi}^{\pi}(u(x)-M)|f(x)|^2dx-\left|\int_{-\pi}^{\pi}v_1(x)f(x)dx\right|^2-\left|\int_{-\pi}^{\pi}v_2(x)f(x)dx\right|^2$$
$$-\left|\int_{-\pi}^{\pi}v_3(x)f(x)dx\right|^2=0.$$

Since each term in the last equalities is non-positive, we obtain the equalities

$$\int_{-\pi}^{\pi}(u(x)-M)|f(x)|^2dx=0,\quad v_k\perp f,\ \ k=1,2,3.$$

From here we conclude that $f\equiv 0$ and this assertion contradicts the condition $||f||=1$. Therefore, $M\notin W(H)$.

Now we prove assertion b) of Theorem 1, that is, the number $M\in\sigma_{\text{ess}}(H)$ is a limit point for the set $W(H)$. Since $M\in\sigma_{\text{ess}}(H)$ according to the Weyl creation [13] there exists orthonormal system of elements $\{g_n\}\subset L_2[-\pi;\pi]$ such that

$$\varphi_n:=Hg_n-Mg_n\to 0,\ \ n\to\infty.$$

Consider the scalar product (φ_n,g_n) and we study this scalar product as $n\to\infty$:

$$(\varphi_n,g_n)=(Hg_n,g_n)-M(g_n,g_n)=(Hg_n,g_n)-M\to 0,\ \ n\to\infty.$$

From this, in turn, it follows that the number M is a limit point for the set $W(H)$. Theorem 1 is completely proved.

In order to use it in our further research, along with the Friedrichs model H, we consider linear, bounded and self-adjoint operators V_k, $k=1,2,3$ in the Hilbert space $L_2[-\pi;\pi]$. By the construction the Friedrichs models H_1, H_2 and H_3 have one-dimensional perturbation and are simpler than the Friedrichs model H.

We present the second result of the paper.

Theorem 2. There are indices $i,j,k\in\{1,2,3\}$ such that

$$W(H_i)\subset W(H_j)\subset W(H_k).$$

Proof. First we describe the set W(H_k) for any index $k \in \{1,2,3\}$. Using the considerations in the proof of the assertion a) of Theorem 1, we get that $M \notin W(H_k)$. By the definition of the numerical range $W(H_k)$ we have

$$\overline{W(H_k)}=[\min\sigma(H_k); M].$$

Using simple considerations, it can be shown that the Friedrichs model H_k has at most one simple eigenvalue lying to the left of the point m and no eigenvalues lying to the right of the point M. Thus, if there is an eigenvalue of the Friedrichs model H_k, we denote it by E_k, and in this case the relation $\min\sigma(H_k) = E_k < m$ is fulfilled. Otherwise, i.e., if there are no eigenvalues of the Friedrichs model H_k, then the equality $\min\sigma(H_k) = m$ holds.

Three case can occur.

First case. Suppose that for any index $k \in \{1,2,3\}$ the Friedrichs model H_k has an eigenvalue lying to the left of point m, and let $E_2 \le E_1 \le E_3$. In this case for indices $i = 3, j = 1, k = 2$ we have $W(H_3) \subset W(H_1) \subset W(H_2)$.

Second case. Suppose that the Friedrichs model H_2 has no eigenvalues lying to the left of point m, and H_1 and H_3 Friedrichs models have eigenvalues lying to the left of point m, and let $E_3 \le E_1 < m$. In this case for the indices $i = 3, j = 1, k = 2$ the inclusions $W(H_2) \subset W(H_1) \subset W(H_3)$ hold.

Third case. Suppose that for any index $k \in \{1,2,3\}$ the Friedrichs model H_k does not have eigenvalues lying to the left of point m, i.e. $\min\sigma(H_k) = m$. In this case, the equalities $W(H_1) = W(H_2) = W(H_3)$ are fulfilled.

Theorem 2 is completely proved.

In order to formulate the next result, we denote the support of the function $v_k(\cdot)$ by $\operatorname{supp}\{v_k\}$, and the Lebesgue measure of the set $\Omega \subset R$ by $\mu(\Omega)$.

We present the third result of the paper.

Theorem 3. If for all indices $i \ne j, i, j = 1,2,3$ the condition

$$\mu(\operatorname{supp}\{v_i\} \cap \operatorname{supp}\{v_j\}) = 0 \qquad (2)$$

holds, then there is an index $k \in \{1,2,3\}$ such that $W(H_k) = W(H)$.

Proof. Assume that the condition (2) is fulfilled for any indices $i \ne j, i, j = 1,2,3$. Then for these indices the equality $I_{ij}(z) = 0$ holds for all $z \in C\backslash[m;M]$ and the function $\Delta(\cdot)$ can be represented as a product of three function

$\Delta(z) = (1 - I_{11}(z))(1 - I_{22}(z))(1 - I_{33}(z))$.

For any index $k \in \{1,2,3\}$ the Friedrichs model H_k has an eigenvalue z lying to the left of point m if and only if $I_{kk}(z) = 1$. Therefore, the number $z \in C\backslash[m;M]$ is an eigenvalue of the Friedrichs model H if and only if this number z is an eigenvalue of the at least one of the Friedrichs models H_1, H_2 and H_3. From this fact we obtain the equality

$$\sigma_{\text{disc}}(H) = \sigma_{\text{disc}}(H_1) \cup \sigma_{\text{disc}}(H_2) \cup \sigma_{\text{disc}}(H_3). \qquad (3)$$

By the definition of the Friedrichs models $H_k, k = 1,2,3$ and by the equality (3) for the spectrum of the Friedrichs model H we obtain

$\sigma(H) = \sigma_{\text{disc}}(H_1) \cup \sigma_{\text{disc}}(H_2) \cup \sigma_{\text{disc}}(H_3) \cup [m;M]$.

From the equality (3) we conclude that

$$\min\sigma(H) = \min\{\min\sigma_{\text{disc}}(H_1), \min\sigma_{\text{disc}}(H_2), \min\sigma_{\text{disc}}(H_3)\}.$$

From the remarks used in the proof of Theorem 2 and using the equality $\overline{W(H)}=[\min\sigma(H); M]$ we obtain the inclusion $W(H_k) \subset W(H)$ for all indices $k \in \{1,2,3\}$. By the definition of the set $W(H_k)$ there exists the index $k \in \{1,2,3\}$ such that $W(H_k) = W(H)$.

Theorem 3 is completely proved.

It should be noted that the class of functions $v_k(\cdot), k = 1,2,3$ satisfying condition (3) is not empty. For example, the parameter functions

$$v_1(x)=\begin{cases}\sin(2x), & \text{if } x\in\left[\frac{\pi}{2};\pi\right];\\ 0, & \text{if } x\in\left[-\pi;\frac{\pi}{2}\right];\end{cases}$$

$$v_2(x)=\begin{cases}\sin(2x), & \text{if } x\in\left[0;\frac{\pi}{2}\right];\\ 0, & \text{if } x\in[-\pi;0]\cup\left[\frac{\pi}{2};\pi\right];\end{cases}$$

$$v_3(x)=\begin{cases}\sin(2x), & \text{if } x\in[-\pi;0];\\ 0, & if\ \ x\in[0;\pi];\end{cases}$$

are real-valued, continuous and linearly independent functions defined on $[-\pi;\pi$ satisfy the condition (3).

Conclusion. In this paper the relation between spectrum and numerical range of the Friedrichs model H corresponding to the Hamiltonian of the system of two particles on a lattice are studied. The Friedrichs model has a rank three perturbation and its Fredholm determinant is constructed. Under natural conditions on the parameter functions $v_1(\cdot), v_2(\cdot)$ and $v_3(\cdot)$ the relations between numerical ranges of the Friedrichs model H and the Friedrichs models H_1, H_2 and H_3 with rank one perturbation are established. The results obtained in the paper are important in determining the essential spectrum of the model operator corresponding to the Hamiltonian of the three-particle system on the lattice, and in finding the number of components that make it up [14,15].

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