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A NEW METHOD FOR NORM OF A HARDY-VOLTERRA OPERATOR

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Abstract. In this paper, we consider the problem of finding the norm of the Hardy-Volterra operator with convolution kernel in weighted Lebesgue spaces**.**
Keywords: Hardy-Volterra operator, weighted Lebesgue space, norm, kernel, weight.

Hardi-Volterra operatori normasi uchun yangi metod

Annotatsiya. Ushbu ishda vaznli Lebeg fazosida berilgan o`rama yadroli Hardi-Volterra operatori normasini topish masalasi qaraladi.
Kalit so`zlar: Hardi-Volterra operatori, vaznli Lebeg fazosi, operator normasi, yadro, vazn.

Новый метод для нормы оператора Харди-Вольтерра

Аннотация. В данной работе рассматривается задача нахождения нормы оператора Харди-Волтерра с ядром свертки в весовом пространстве Лебега.
Ключевая слова: Операторы Харди-Волтерра, весовые пространства Лебега, норма оператора, ядро, вес

.

1. Introduction

The weighted Lebesgue space $L_{s,w}$, with $s \geq 1$ and a weight function (nonegative measurable function) w on $(0,\infty)$, is defined as

$$L_{s,w} = \left\{ f:\ \|f\|_{s,w} := \left(\int_0^\infty |f(x)|^s\, w(x)dx \right)^{\frac{1}{s}} < \infty \right\}.$$

Let $1 \leq p, q < \infty$ and u, v be weight functions. Let us consider the following Hardy-Volterra integral operator:

$$H: L_{p,v} \to L_{q,u}\ , \qquad H(f) := \int_0^x k(x,t)f(t)dt, \qquad (1)$$

where $k(x,t)$ is a kernel – measurable function defined on $(0;\infty)\times(0;\infty)-$ of the operator. The problem of discovering necessary and sufficient conditions for the boundedness of H, i.e.,

$$\|Hf\|_{q,u} \leq C\|f\|_{p,v} \qquad (2)$$

and obtaining exact values or two-sided estimates for its norm – the least constant C for which (2) holds – have been investigated by many authors (see [1-6]).

In 1925, G.H. Hardy [1] derived inequality (2) in the case $p = q > 1$ and $u(x) = \frac{1}{x^p}$, $v(x) = 1$ for the Hardy operator

$$(Hf)(x) := \int_0^x f(t)dt$$

with the exact value for the norm $\|H\|_{L_{p,u}\to L_p} = \frac{p}{p-1}$, namely, inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \qquad (3)$$

which is called the classical Hardy inequality in L_p. Later, in 1928, he proved the "weighted" modification of (2) for $u(x) = \frac{1}{x^{p-\alpha}}$, $v(x) = x^\alpha$ in (2), i.e., the inequality

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x} f(t)dt\right)^p x^{\alpha}dx$$

$$\leq\left(\frac{p}{p-\alpha-1}\right)^p\int_0^{\infty} f(x)^p x^{\alpha}dx \qquad (4)$$

with $p>1$ and $\alpha<p-1-$ for all measurable nonnegative functions f (see [1]), here the constant $C=\|H\|_{L_{p,v}\to L_{p,u}}=\frac{p}{p-\alpha-1}$. In the case when one of the parameters p and q is equal to 1 or ∞, there is a general result presenting the exact value of the best constants in (1) (see [3]).

The problem becomes more diffucult with an integral operator (1) with a nonconstant kernel $k(x,t)$. Martin-Reyes and Sawyer [11] and Prokhorov and Stepanov [9,12,13] considered (1) with the kernel $k(x,t)=(x-t)^{r-1}$ $,r>0$. S. Bloom and R. Kerman [14] and R. Oinarov [15] gave equivalent conditions for boundedness of (1) for kernels $k(x,t)$ is a continuous nonnegative function increasing in the first argument, decreasing in the second argument and satisfying the condition: there exists a number $h>1$ such that $k(x,s)\leq h\big(k(x,t)+k(t,s)\big)$ for all $s\leq t\leq x$. It was proven that the conditions

$$A_1=\sup_{x\in(0,+\infty)}\left(\int_x^{\infty}k^q(t,x)u(t)dt\right)^{\frac{1}{q}}\left(\int_0^{x}v^{1-p'}(t)dt\right)^{\frac{1}{p'}}<\infty\ ,$$

$$A_2=\sup_{x\in(0,\infty)}\left(\int_x^{\infty}u(t)dt\right)^{\frac{1}{q}}\left(\int_0^{x}k^{p'}(x,t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}}<\infty$$

are equivalent to satisfying inequality (2). Kufner, Kuliev and Oinarov in [16] and Kufner, Kuliev and Persson in [17] also considered the boundedness of (1) with the kernel $\sum_{i=1}^{n}a_i(x)b_i(t)$. In [10], Drábek, Kuliev and Marletta also gave an estimate for the norm for a special choice of the kernel $k(x,t)=x-t$, which appears from the fourth-order nonlinear Shturm-Liouville problem, where they significantly used these estimates. In 2021, Kalybay and Baiarystanov [18] proved the estimates

$$A\leq C\leq(h+1)^3p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A,$$

where $A=max\{A_1,A_2\}<\infty$. Recently, Kuliev, Kulieva and Eshimova in [6] also obtained the estimate $A\leq C\leq X$, where X is a positive solution of the equation $X^{q'}-hq^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}X=hq^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'}$, where $q'=\frac{q}{q-1}$. In this paper, we give a new proof for finding the norm of the Hardy-Volterra operator (1) in the case $p=q=2$ and $u(x)=x^{\varepsilon-2n-2}$ $,v(x)=x^{\varepsilon}$, $\varepsilon<1$ and the kernel

$$k(x,t)=(x-t)^n\ ,\ n\in\boldsymbol{Z}_+,\qquad \text{i.e.,}$$

$H\colon L_{2,x^{\varepsilon}}\to L_{2,x^{\varepsilon-2n-2}}$,

$$H(f):=\int_0^{x}(x-t)^n f(t)dt. \qquad (5)$$

The paper is organized as follows: the first section is the introduction. In the second section, we give our main results, the proofs of which are given in the next section.

1. Main results

The problem of finding the norms of linear operators in Hilbert spaces is also frequently encountered in mathematical physics. Unfortunately, such questions cannot always be answered positively. In this section, we give a new method for finding the norm of the Hardy-Volterra operator (5).

Our main result of the work is

Theorem 1. *Let* $p=q=2$, $\varepsilon<1$ *and* $n\in\boldsymbol{Z}_+$. *Then, (5) is a bounded operator with the norm*

$$\|H\|_{L_{2,x^{\varepsilon}}\to L_{2,x^{\varepsilon-2n-2}}}=B\left(\frac{1-\varepsilon}{2},n+1\right), \qquad (6)$$

where B is the Beta function.

Remark 1. a) The boundedness of (5) can be written as

$$\int_0^\infty \left(\int_0^x (x-t)^n f(t)dt \right)^2 x^{\varepsilon-2n-2}dx \le C_n^2 \int_0^\infty f^2(x)x^\varepsilon dx, \qquad (7)$$

which is also called Hardy-type inequality. In particular, when $n = 0$, we have (4) in the case $p = 2$

$$\int_0^\infty \left(\int_0^t f(s)ds \right)^2 t^{\varepsilon-2}dt \le \left(\frac{2}{1-\varepsilon}\right)^2 \int_0^\infty f^2(t)t^\varepsilon dt, \qquad (8)$$

where

$$C_0 = B\left(\frac{1-\varepsilon}{2}, 1\right) = \frac{2}{1-\varepsilon} \quad , \qquad \text{with} \quad \varepsilon < 1.$$

b) If $\varepsilon = 0$ and $n = 0$, then from (7), we obtain the classical Hardy inequality (3) with the sharp constant 4.

1. Proof of main result

Proof of Theorem 1.

In the proof of the theorem, we use the method of mathematical induction, and we prove the theorem in 4 steps.

Step 1. (The case $n = 0$.) In this case, inequality (7) takes (8), i.e., the norm is equal to $\frac{2}{1-\varepsilon} = B\left(\frac{1-\varepsilon}{2}, 1\right)$.

Step 2. Next, we suppose that the theorem is true for $k = 1,2, \dots, n-1$, i.e.,

$$\int_0^\infty \left(\int_0^x (x-t)^k f(t)dt \right)^2 x^{\varepsilon-2k-2}dx \le C_k^2 \int_0^\infty f^2(x)x^\varepsilon dx \qquad (9)$$

is valid with the best constant $C_k = B\left(\frac{1-\varepsilon}{2}, k+1\right)$. Now, we prove (7) for $k = n$.

First, we denote the left side of (7) by I_n and perform the following operations:

$$I_n = \int_0^\infty \left(\int_0^x (x-t)^n f(t)dt \right)^2 x^{\varepsilon-2n-2}dx.$$

Represent the squar of the interior integral as

$$\left(\int_0^x (x-t)^n f(t)dt \right)^2 = 2\int_0^x (x-t)^n f(t) \left(\int_0^t (x-s)^n f(s)ds \right) dt =$$

$$= 2\sum_{k=0}^n C_n^k \int_0^x (-t)^{2n-k} f(t) \left(\int_0^t (-s)^k f(s)ds \right) dt. \qquad (10)$$

Here, we used Newton's binomial formulae as

$$(x-s)^n = [(x-t)+(t-s)]^n = \sum_{k=0}^n C_n^k (x-t)^{n-k}(t-s)^k.$$

Putting the representation of the inner integral (10) into the expression of I_n, we getobtain the following:

$$I_n = 2\sum_{k=0}^n C_n^k \int_0^\infty \left[\int_0^x (x-t)^{2n-k} f(t) \left(\int_0^t (t-s)^k f(s)ds \right) dt \right] x^{\varepsilon-2n-2}dx.$$

Then, applying Foubini's theorem and Hölder's inequality, we obtain

$$I_n = 2\sum_{k=0}^{n} C_n^k \int_0^{\infty} f(t)\left(\int_0^t (t-s)^k f(s)ds\right)\left[\int_t^{\infty} (x-t)^{2n-k}x^{\varepsilon-2n-2}dx\right]dt \le$$

$$\le 2\sum_{k=0}^{n} C_n^k \left(\int_0^{\infty} f(t)t^{\varepsilon}dt\right)^{\frac{1}{2}}\left(\int_0^{\infty} t^{-\varepsilon}\left(\int_0^t (t-s)^k f(s)ds\right)^2\left[\int_t^{\infty} (x-t)^{2n-k}x^{\varepsilon-2n-2}dx\right]^2 dt\right)^{\frac{1}{2}}$$

$$= 2\|f\|_{L_{p,x^{\varepsilon}}}\sum_{k=0}^{n} C_n^k \cdot J_{k,n}{}^{\frac{1}{2}}.$$

In the inner integral of $J_{k,n}$ with the square bracket, we chage the variables first as $x = t \cdot s$ and then as $\tau = \frac{1}{s}$ and obtain

$$\int_t^{\infty} (x-t)^{2n-k}x^{\varepsilon-2n-2}dx = t^{\varepsilon-k-1}\int_1^{\infty} (s-1)^{2n-k}s^{\varepsilon-2n-2}ds =$$

$$= t^{\varepsilon-k-1}\int_0^1 (1-\tau)^{2n-k}\tau^{k-\varepsilon}d\tau = t^{\varepsilon-k-1}B(k-\varepsilon+1,2n-k+1).$$

Then, $J_{k,n}$ can be written as

$$J_{k,n} = \int_0^{\infty} t^{-\varepsilon}\left(\int_0^t (t-s)^k f(s)ds\right)^2 [t^{\varepsilon-k-1}B(k-\varepsilon+1,2n-k+1)]^2 dt =$$

$$= B^2(k-\varepsilon+1,2n-k+1)\left(\int_0^{\infty}\left(\int_0^t (t-s)^k f(s)ds\right)^2 t^{\varepsilon-2k-2}dt\right) =$$

$$= B^2(k-\varepsilon+1,2n-k+1)\cdot I_k.$$

Eventually, we have the estimate for I_n:

$$I_n \le 2\|f\|_{L_{p,x^{\varepsilon}}}\sum_{k=0}^{n} C_n^k B(k-\varepsilon+1,2n-k+1)\cdot I_k^{\frac{1}{2}}.$$

On the right-hand side of the estimate for the integral I_n, we rewrite the sum separately from the last n-th term:

$$I_n \le 2\|f\|_{L_{p,x^{\varepsilon}}}\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1,2n-k+1)\cdot I_k^{\frac{1}{2}} + 2\|f\|_{L_{p,x^{\varepsilon}}}B(n-\varepsilon+1,n+1)\cdot I_n^{\frac{1}{2}}.$$

By the assumption, we replace $I_k^{\frac{1}{2}}$ with its upper estimate $C_k\|f\|_{L_{p,x^{\varepsilon}}}$, where

$C_k = B\left(\frac{1-\varepsilon}{2}, k+1\right)$ and then

$$I_n \le 2\|f\|_{L_{p,x^{\varepsilon}}}\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1,2n-k+1)\cdot C_k\|f\| + 2\|f\|_{L_{p,x^{\varepsilon}}}B(n-\varepsilon+1,n+1)\cdot I_n^{\frac{1}{2}}$$

$$= 2\|f\|_{L_{p,x^{\varepsilon}}}{}^2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1,2n-k+1)\cdot B\left(\frac{1-\varepsilon}{2},k+1\right) + 2\|f\|B(n-\varepsilon+1,n+1)\cdot I_n^{\frac{1}{2}}$$

If $\frac{I^{\frac{1}{2}}}{\|f\|_{L_{p,x^{\varepsilon}}}} = X$, then the last estimate takes the form

$$X^2 - 2B(n-\varepsilon+1, n+1)X - 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1, 2n-k+1)B\left(\frac{1-\varepsilon}{2}, k+1\right) \le 0,$$

If we solve the following quadratic equation

$$X^2 - 2B(n-\varepsilon+1, n+1)X - 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1, 2n-k+1)B\left(\frac{1-\varepsilon}{2}, k+1\right) = 0,$$

then we have

$$C_n \le X = B(n-\varepsilon+1, n+1)$$

$$+\left[B^2(n-\varepsilon+1, n+1) + 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1, 2n-k+1)B\left(\frac{1-\varepsilon}{2}, k+1\right)\right]^{\frac{1}{2}}.$$

Step 3. Now, we find a lower estimate for the constant C_n. Here, we suppose that inequality (7) is true. We choose the test function

$$f_\tau(t) = \begin{cases} t^\alpha, & 0 < t < \tau \\ 0, & t > \tau \end{cases},$$

$$\alpha > -\frac{1+\varepsilon}{2}. \qquad (11)$$

The right-hand side of (7) is

$$\int_0^\infty f_\tau^2(t) \cdot t^\varepsilon dt = \int_0^\tau t^{2\alpha+\varepsilon} dt = \frac{\tau^{2\alpha+\varepsilon+1}}{2\alpha+\varepsilon+1},$$

The left-hand side of (7) is

$$\int_0^\infty \left(\int_0^x f_\tau(t)(x-t)^n dt\right)^2 x^{\varepsilon-2n-2} dx \ge \int_0^\tau \left(\int_0^x (x-t)^n \cdot t^\alpha dt\right)^2 x^{\varepsilon-2n-2} dx =$$

$$= B^2(\alpha+1, n+1) \cdot \frac{\tau^{2\alpha+\varepsilon+1}}{2\alpha+\varepsilon+1} \quad .$$

Then,

$$B^2(\alpha+1, n+1) \le C_n^2,$$

The supremum of

$$\sup_{\alpha > -\frac{1+\varepsilon}{2}} B^2(\alpha+1, n+1) = B^2\left(\tfrac{1-\varepsilon}{2}, n+1\right),$$

i.e.,

$$B^2\left(\tfrac{1-\varepsilon}{2}, n+1\right) \le C_n^2.$$

Step 4. Now, we prove that the upper and lower estimates of C_n are equal. Consider the equality

$$B\left(\tfrac{1-\varepsilon}{2}, n+1\right) = B(n-\varepsilon+1, n+1) +$$

$$+\left[B^2(n-\varepsilon+1, n+1) + 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1, 2n-k+1)B\left(\frac{1-\varepsilon}{2}, k+1\right)\right]^{\frac{1}{2}}$$

or

$$B\left(\tfrac{1-\varepsilon}{2}, n+1\right) - B(n-\varepsilon+1, n+1) =$$

$$= \left[B^2(n-\varepsilon+1,n+1) + 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1,2n-k+1) B\left(\frac{1-\varepsilon}{2},k+1\right)\right]^{\frac{1}{2}}.$$

Then, both sides are placed in the square to obtain

$$B^2\left(\frac{1-\varepsilon}{2},n+1\right) - 2B\left(\frac{1-\varepsilon}{2},n+1\right)B(n-\varepsilon+1,n+1) + B^2(n-\varepsilon+1,n+1) =$$

$$= B^2(n-\varepsilon+1,n+1) + 2\sum_{k=0}^{n-1} C_n^k B(k-\varepsilon+1,2n-k+1) B\left(\frac{1-\varepsilon}{2},k+1\right).$$

Simplifying this expression, we obtain the following equality:

$$B^2\left(\frac{1-\varepsilon}{2},n+1\right) = 2\sum_{k=0}^{n} C_n^k B(k-\varepsilon+1,2n-k+1) B\left(\frac{1-\varepsilon}{2},k+1\right) =$$

$$= 2\sum_{k=0}^{n} C_n^k \left(\int_0^1 x^{k-\varepsilon}(1-x)^{2n-k}dx\right)\left(\int_0^1 y^{\frac{1-\varepsilon}{2}-1}(1-y)^k dy\right) =$$

$$= 2\sum_{k=0}^{n} C_n^k \int_0^1\int_0^1 x^{k-\varepsilon}(1-x)^{2n-k} y^{\frac{1-\varepsilon}{2}-1}(1-y)^k dxdy =$$

$$= 2\int_0^1\int_0^1 x^{-\varepsilon}(1-x)^n y^{\frac{1-\varepsilon}{2}-1}\left(\sum_{k=0}^{n} C_n^k (1-x)^{n-k}(x(1-y))^k\right)dxdy.$$

Applying Newton's binomial formulae to the inner bracket, we have

$$= 2\int_0^1\int_0^1 x^{-\varepsilon}(1-x)^n y^{\frac{1-\varepsilon}{2}-1}(1-x+x-xy)^n dxdy =$$

$$= 2\int_0^1\int_0^1 x^{-\varepsilon}(1-x)^n y^{\frac{1-\varepsilon}{2}-1}(1-xy)^n dxdy =$$

$$= 2\int_0^1\int_0^1 (xy)^{\frac{1-\varepsilon}{2}-1}(1-xy)^n x^{\frac{1-\varepsilon}{2}}(1-x)^n dxdy.$$

After changing the variables as $xy = t,\ xdy = dt,\ dy = \frac{1}{x}dt$, and $0 \le t \le x$, we obtain

$$= 2\int_0^1 \left(\int_0^x t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx =$$

$$= \int_0^1 \left(\int_0^x t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx +$$

$$+ \int_0^1 \left(\int_0^x t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx.$$

We apply Fubini's theorem to the second term of the sum

$$= \int_0^1 \left(\int_0^x t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx + \int_0^1 \left(\int_t^1 x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx\right) t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt =$$

$$= \int_0^1 \left(\int_0^x t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx + \int_0^1 \left(\int_x^t t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt\right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx$$

Combining the two integrals, we bring back Beta functions

$$= \int_0^1 \left(\int_0^1 t^{\frac{1-\varepsilon}{2}-1}(1-t)^n dt \right) x^{\frac{1-\varepsilon}{2}-1}(1-x)^n dx = B^2\left(\frac{1-\varepsilon}{2}, n+1\right).$$

Therefore, the upper and lower estimates for the constant C_n are equal. In conclusion, the best constant for the Hardy-type inequality (10) or the norm of operator (6) is of the form

$$C_n = \|H\|_{L_{2,x^\varepsilon}\to L_{2,x^{\varepsilon-2n-2}}} = B\left(\frac{1-\varepsilon}{2}, n+1\right).$$

Theorem is proved.

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