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NONLINEAR DIFFERENTIAL EQUATION WITH GERASIMOV-CAPUTO TYPE FRACTIONAL OPERATOR AND DEGENERATOR

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Abstract. In this article a nonlinear initial value problem for a Gerasimov-Caputo type fractional differential equation with degeneration is considered in the case of $0<\alpha\leq 1$. The right-hand side of the equation consists product of two nonlinear functions. The solution of this fractional differential equation is studied in the class of regular functions. A nonlinear integral equation is obtained by the Mittag-Leffler function. In proof of one valued solvability of the problem the method of successive approximations in combination with the method of compressing mapping is applied. The continuous dependence of solution of the problem on initial data also is studied.

Keywords: initial value problem, Gerasimov-Caputo fractional equation, degeneration, product of two nonlinear functions, Mittag-Leffler function, method of compressing mapping.

Gerasimov-Kaputo turidagi kasr tartibli nochiziqli differensial tenglamalar

Annotatsiya. Ushbu maqolada aynigan Gerasimov-Kaputo turidagi kasr tartibli differensial tenglamalar uchun bo‘lganda chiziqsiz boshlang‘ich masala o‘rganildi. Tenglamaning o‘ng tomoni ikkita chiziqsiz funksiyalarning ko‘paytmasidan iborat. Ushbu kasr tartibli differensial tenglamaning echimi regulyar funksiyalar sinfida qaralgan. Mittag-Leffler funksiyasi yordamida chiziqsiz integral tenglama hosil qilindi. Yagona echimning mavjudligi ketma-ket yaqinlashish usuliga qisqartirib akslantirish usulini qo‘llash orqali isbot qilindi. Shuningdek, masala echimining boshlang‘ich ma'lumotlarga nisbatan usluksiz ekanligi isbotlandi.

Kalit so‘zlar: boshlang‘ich masala, Gerasimov-Kaputo kasr tenglamasi, aynish, ikkita chiziqsiz funksiyalarning ko‘paytmasi, Mittag-Leffler funksiyasi, qisqartirib aks ettirish usuli.

Нелинейное дифференциальное уравнение с дробным оператором типа Герасимова-Капуто и вырождением

Аннотация. В статье рассматривается нелинейная начальная задача для дробного дифференциального уравнения типа Герасимова-Капуто с вырождением в случае $0<\alpha\leq 1$. Правая часть уравнения состоит из произведения двух нелинейных функций. Решение этого дробного дифференциального уравнения изучается в классе регулярных функций. Нелинейное интегральное уравнение получено с помощью функции Миттаг-Леффлера. Для доказательства однозначной разрешимости задачи применяется метод

последовательных приближений в сочетании с методом сжимающих отображений. Изучается также непрерывная зависимость решения задачи от начальных данных.

Ключевые слова: начальная задача, дробное уравнение Герасимова-Капуто, вырождение, произведение двух нелинейных функций, функция Миттаг-Леффлера, метод сжимающего отображения.

Introduction. Statement of the problem. Functional differential equations containing a product of two nonlinear functions arise when solving problems of nonlinear mechanics and nonlinear optimal control. Of theoretical interest is the study of nonlocal problems for functional differential equations. By today a lot of publications of studying nonlocal problems for differential equations, describing many natural and practical processes, are appeared [1-11]. The interest on the study of nonlocal problems for the fractional differential equations is high by the mathematicians (see, for example, [12-24]).

On the segment $[0,T]$ the fractional differential equation of the following form is considered

$$ {}_C D_{0t}^{\alpha} x(t) + t^{\beta} x(t) = f(t, x(t))\, g(t, x(t)) \tag{1} $$

with initial value condition

$$ x(0) = \varphi_0 = const, \tag{2} $$

where β, T are given positive real numbers, for $0 < \alpha \le 1$

$$ {}_C D_{0t}^{\alpha} \eta(t) = J_{0t}^{1-\alpha} \eta'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\eta'(s)}{(t-s)^{\alpha}} ds, \quad {}_C D_{0t}^{1} \eta(t) = \eta'(t), \quad t \in (0,T) $$

is Gerasimov-Caputo type fractional operator,

$$ J_{0t}^{\alpha} \eta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\eta(s)\, ds}{(t-s)^{1-\alpha}}, \quad t \in (0,T) $$

is Riemann-Liouville integral operator, $f(t, x(t)) \in C([0,T] \times X, R)$, $g(t, x(t)) \in C([0,T] \times X, R)$, X is closed set in X.

Problem. To find the function $x(t)$, which satisfies partial differential equation (1), initial value condition (2).

Construction of the solution. The equation (1) rewrite as

$$ {}_C D_{0t}^{\alpha} x(t) = -t^{\beta} x(t) + f(t, x(t))\, g(t, x(t)). \tag{3} $$

It is well known, that the two-parametric Mittag-Leffler function defined as

$$ E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad z, \alpha, \beta \in, \quad \mathrm{Re}(\alpha) > 0. $$

The generalized Mittag-Lefler (Kilbas-Saigo) type function was defined for real $\alpha, m, l \in R$ **and complex** $l \in$ **by Kilbas and Saigo in the following form**

$$ E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \ c_0 < 1, \ c_k = \prod_{j=1}^{k-1} \frac{\Gamma(\alpha[jm + l] + 1)}{\Gamma(\alpha[jm + l + 1] + 1)}, \quad k = 1, 2, \ldots $$

These functions belong to the class of entire functions in the complex plane.

As an analog of the initial value problem (3), (2), we consider the following auxiliary equation

$$ {}_C D_{0t}^{\alpha} x(t) = -t^{\beta} x(t) + h(t), \quad x(0) = \varphi_0, \tag{4} $$

where $\beta, \varphi_0 \in R, \ h(t) \in C[0,T]$ is a given continuous function.

Let $\gamma \in [0,1)$. Then we consider the class of following functions ([17], p. 4, 205):

$$ C_{\gamma}[0;T] = \left\{ h(t) : t^{\gamma} h(t) \in C[0;T] \right\}, $$

$$ C_{\gamma}^{\alpha}[0;T] = \left\{ h(t) \in C[0;T] : {}_C D_{0t}^{\alpha} h(t) \in C_{\gamma}[0;T] \right\}. $$

Lemma 1. *Let* $\gamma \in [0;\alpha],\ \beta \geq 0$. *Then for all* $h(t) \in C_\gamma[0;T]$ *there exists a unique solution* $x(t) \in C_\gamma^\alpha[0;T]$ *of the Cauchy problem* (4). *This solution has the following form*

$$x(t) = \varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(t^{\alpha+\beta}\right) + \int_0^t K(t,s)h(s)ds, \tag{5}$$

where

$$K(t,s) = \sum_{i=1}^{\infty} K_i(t,s), \tag{6}$$

$$K_0(t,s) = \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1},\ K_i(t,s) = \frac{1}{\Gamma(\alpha)}\int_s^t \theta^\beta (t-\theta)^{\alpha-1} K_{i-1}(\theta,s)d\theta,\ i = 1,2,\ldots \tag{7}$$

$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(t^{\alpha+\beta}\right)$ *is Kilbas-Saigo function.*

Proof. We consider the inhomogeneous problem (4) and the solution of this problem we seek as the sum of two functions

$$x(t) = v(t) + w(t), \tag{8}$$

where the functions $v(t)$ and $w(t)$, respectively, are solutions of the following two problems:

$${}_C D_{0t}^\alpha v(t) = -t^\beta v(t),\quad v(0) = \varphi_0, \tag{9}$$

$${}_C D_{0t}^\alpha w(t) = -t^\beta w(t) + h(t),\quad w(0) = 0, \tag{10}$$

It is proved in [17] (see, theorem 4.4 there) that, the problem (9) has a unique solution $v(t) \in C_\gamma^\alpha[0;T]$ of the form

$$v(t) = \varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right). \tag{11}$$

We study the problem (10). The problem is equivalent to the one valued solvability of Volterra integral equation of the second kind

$$w(t) = -J_{0t}^\alpha\left(t^\beta w(t)\right) + J_{0t}^\alpha h(t),\quad t \in (0;T). \tag{12}$$

We apply the method of successive approximations to solve the integral equation (12). In this order we consider the iteration process

$$\begin{cases} w^0(t) = J_{0t}^\alpha h(t), \\ w^k(t) = w^0(t) - J_{0t}^\alpha\left(t^\beta w^{k-1}(t)\right),\ k = 1,2,3,\ldots \end{cases}$$

Hence, it is not difficult to obtain that the desired solution of the problem (10) has the form

$$w(t) = \int_0^t K(t,s)h(s)ds, \tag{13}$$

where the kernel $K(t,s)$ defines by the formulas (6) and (7). According to the formula (8), from the representations (12) and (13) we have the solution (5). Lemma 1 is proved.

Lemma 2 ([13], pp. 9)**.** *For every* $\alpha \in [0,1],\ m > 0$ *and* $t \geq 0$ *the estimate is true*

$$\frac{1}{1+\Gamma(1-\alpha)t} \leq E_{\alpha,m,m-1}(-t) \leq \frac{1}{1+\dfrac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)}t}.$$

Lemma 3 ([14], pp. 136)**.** *Let* $\alpha < 2,\ z \in \mathbb{C}$, δ *be real constant and* σ *be fixed number from the interval* $\left(\frac{\pi\alpha}{2}, \min\{\pi, \pi\alpha\}\right)$. *Then for* $|\arg z| \leq \sigma$ *and* $|z| \geq 0$ *the estimate is true*

$$\left|E_{\alpha,\mu}(z)\right| \leq M_1(1+|z|)^{\frac{1-\delta}{\alpha}} e^{\mathrm{Re}\, z^{\frac{1}{\alpha}}} + \frac{M_2}{1+|z|},$$

where M_1 *and* M_2 *are constants, not depending from* z.

Lemma 4. *For the case of* $\gamma \in [0;\alpha],\ \beta \geq 0$ *there holds the following estimate*

$$|K(t,s)| \leq (t-s)^{\alpha-1} M_3, \tag{14}$$

where $M_3 = const$.

Proof. From the presentations (6) and (7) we obtain that there hold the following estimates

$$|K(t,s)| \leq |K_0(t,s)| + |K_1(t,s)| + |K_2(t,s)| + ... |K_k(t,s)| + ..., \tag{15}$$

$$|K_0(t,s)| \leq \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, \tag{16}$$

$$|K_1(t,s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^t \theta^\beta (t-\theta)^{\alpha-1} |K_0(\theta,s)| d\theta \leq$$

$$\leq \frac{1}{\Gamma^2(\alpha)} \int_s^t \theta^\beta (t-\theta)^{\alpha-1} (\theta-s)^{\alpha-1} d\theta \leq \frac{t^\beta}{\Gamma^2(\alpha)} \int_s^t (t-\theta)^{\alpha-1} (\theta-s)^{\alpha-1} d\theta. \tag{17}$$

We use the following known formula

$$\int_s^t (t-\theta)^{\rho-1} (\theta-s)^{\sigma-1} d\theta = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)} (t-s)^{\rho+\sigma-1}. \tag{18}$$

Then, taking $\rho = \sigma = \alpha$ into account in (18), from the estimate (17) we derive

$$|K_1(t,s)| \leq \frac{t^\beta}{\Gamma^2(\alpha)} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} (t-s)^{2\alpha-1} = \frac{1}{\Gamma(2\alpha)} t^\beta (t-s)^{2\alpha-1}. \tag{19}$$

Similarly, we estimate the next kernel $K_2(t,s)$:

$$|K_2(t,s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^t \theta^\beta (t-\theta)^{\alpha-1} |K_1(\theta,s)| d\theta \leq$$

$$\leq \frac{1}{\Gamma^2(\alpha)} \frac{1}{\Gamma(2\alpha)} \int_s^t \theta^{2\beta} (t-\theta)^{\alpha-1} (\theta-s)^{2\alpha-1} d\theta \leq \frac{t^{2\beta}}{\Gamma(\alpha)\Gamma(2\alpha)} \int_s^t (t-\theta)^{\alpha-1} (\theta-s)^{2\alpha-1} d\theta.$$

Hence, taking (18) and (19) into account, for $\rho = \alpha,\ \sigma = 2\alpha$ we derive

$$|K_2(t,s)| \leq \frac{1}{\Gamma(3\alpha)} t^{2\beta} (t-s)^{3\alpha-1}. \tag{20}$$

By the induction method one can obtain that

$$|K_k(t,s)| \leq \frac{\mu_n^{2k}}{\Gamma((k+1)\alpha)} t^{k\beta} (t-s)^{(k+1)\alpha-1}. \tag{21}$$

Taking estimates (16), (19)-(21) into account, for (15) we have

$$|K(t,s)| \leq \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{1}{\Gamma(2\alpha)} t^\beta (t-s)^{2\alpha-1} + \frac{1}{\Gamma(3\alpha)} t^{2\beta} (t-s)^{3\alpha-1} +$$

$$+ ... + \frac{1}{\Gamma(k\alpha+\alpha)} t^{k\beta} (t-s)^{(k+1)\alpha-1} + ... =$$

$$= (t-s)^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(2\alpha)} t^\beta (t-s)^\alpha + \frac{1}{\Gamma(3\alpha)} t^{2\beta} (t-s)^{2\alpha} + \right.$$

$$\left. + ... + \frac{\mu_n^{2k}}{\Gamma(\alpha(k+1))} t^{k\beta} (t-s)^{k\alpha} + ... \right] \leq (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(t^\beta (t-s)^\alpha\right).$$

According to the Lemmas 2 and 3, by putting $z = \mu_n^2 t^\beta (t-s)^\alpha,\ \delta = \alpha,$ we have

$$\left| E_{\alpha,\alpha}\left(t^\beta (t-s)^\alpha\right) \right| \leq M_1 \left[1 + t^\beta (t-s)^\alpha \right]^{\frac{1-\alpha}{\alpha}} e^{\left(t^\beta (t-s)^\alpha\right)^{\frac{1}{\alpha}}} + \frac{M_2}{1 + t^\beta (t-s)^\alpha} \leq M_3.$$

From the last two estimates, we obtain the estimate (14). The Lemma 4 is proved.

Using the Lemmas 1 and 4, we write the solution of the differential equation (3) with initial value condition (2) as

$$x(t)=\varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right)+\int_0^t K(t,s)\,f(s,x(s))\,g(s,x(s))\,ds, \tag{22}$$

where kernel $K(t,s)$ is defined by the formulas (6) and (7).

Main results. The right-hand side of the integral equation we write as a nonlinear integral operator

$$x(t)=J(t;x(t))\equiv\varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right)+\int_0^t K(t,s)\,f(s,x(s))\,g(s,x(s))\,ds, \tag{23}$$

where

$$J(t;x(t)):C([0,T];R)\to C([0,T];R).$$

Theorem 1. Let the following conditions are fulfilled:

1). $M_f=\max\limits_{0\le t\le T}\left|f(t,x(t))\right|<\infty,\quad 0<M_f=const;$

2). $M_g=\max\limits_{0\le t\le T}\left|g(t,x(t))\right|<\infty,\quad 0<M_g=const;$

3). For all $t\in[0,T]$, $x\in X$ there exist such functions $0<A_f(t)\in C\left([0,T],R\right),\quad 0<B_f(t)\in C\left([0,T],R\right),$ that

$$\left|f(t,x(t))\right|\le A_f(t)\left|x(t)\right|+B_f(t);$$

4). For all $t\in[0,T]$, $x\in X$ there exist such functions $0<A_g(t)\in C\left([0,T],R\right),\quad 0<B_g(t)\in C\left([0,T],R\right),$ that

$$\left|g(t,x(t))\right|\le A_g(t)\left|x(t)\right|+B_g(t).$$

5). $\chi_1=\frac{1}{2}M_4\left[M_f\left\|A_g(t)\right\|_{C[0,T]}+M_g\left\|A_f(t)\right\|_{C[0,T]}\right]<1,\quad 0<M_4=const.$

Then, if the solution of initial value problem (1), (2) exists, there holds the following estimate

$$\left\|x(t)\right\|_{C[0,T]}\le\frac{\chi_2}{1-\chi_1},$$

where

$$\chi_2=M_3\left|\varphi_0\right|+\frac{1}{2}M_4\left[M_f\left\|B_g(t)\right\|_{C[0,T]}+M_g\left\|B_f(t)\right\|_{C[0,T]}\right].$$

Proof. Taking into account the conditions of the theorem, we have

$$\left\|x(t)\right\|_{C[0,T]}\le\left|\varphi_0\right|\max_{0\le t\le T}\left|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right)\right|+$$

$$+\frac{1}{2}\max_{0\le t\le T}\int_0^t\left|K(t,s)\right|\left[\left|f(s,x(s))\right|\left|g(s,x(s))\right|+\left|f(s,x(s))\right|\left|g(s,x(s))\right|\right]ds\le$$

$$\le M_3\left|\varphi_0\right|+\frac{1}{2}\max_{0\le t\le T}\int_0^t\left|K(t,s)\right|ds\left[\left\|f(t,x(t))\right\|_{C[0,T]}\left(\left\|A_g(t)\right\|_{C[0,T]}\left\|x(t)\right\|_{C[0,T]}+\left\|B_g(t)\right\|_{C[0,T]}\right)+\right.$$

$$\left.+\left\|g(t,x(t))\right\|_{C[0,T]}\left(\left\|A_f(t)\right\|_{C[0,T]}\left\|x(t)\right\|_{C[0,T]}+\left\|B_f(t)\right\|_{C[0,T]}\right)\right]\le$$

$$\le M_3\left|\varphi_0\right|+\frac{1}{2}M_4\left[M_f\left(\left\|A_g(t)\right\|_{C[0,T]}\left\|x(t)\right\|_{C[0,T]}+\left\|B_g(t)\right\|_{C[0,T]}\right)+\right.$$

$$\left.+M_g\left(\left\|A_f(t)\right\|_{C[0,T]}\left\|x(t)\right\|_{C[0,T]}+\left\|B_f(t)\right\|_{C[0,T]}\right)\right]\le\chi_1\left\|x(t)\right\|_{C[0,T]}+\chi_2,$$

where

$$M_4 = \max_{0\le t\le T}\int_0^t \left| K(t,s)\right| ds \le \max_{0\le t\le T}\int_0^t (t-s)^{\alpha-1}\left| E_{\alpha,\alpha}\left(t^{\beta}(t-s)^{\alpha}\right)\right| ds \le M_3 \max_{0\le t\le T}\frac{t^{\alpha}}{\alpha},$$

$$\chi_1 = \frac{1}{2}M_4\left[M_f \left\| A_g(t)\right\|_{C[0,T]} + M_g \left\| A_f(t)\right\|_{C[0,T]}\right],$$

$$\chi_2 = M_3\left|\varphi_0\right| + \frac{1}{2}M_4\left[M_f \left\| B_g(t)\right\|_{C[0,T]} + M_g \left\| B_f(t)\right\|_{C[0,T]}\right].$$

Hence, we obtain

$$\left\| x(t)\right\|_{C[0,T]} \le \frac{\chi_2}{1-\chi_1}, \quad \chi_1 < 1.$$

Theorem 2. Let the following conditions are fulfilled:

1). $M_f = \max_{0\le t\le T}\left| f(t,x(t))\right| < \infty, \quad 0 < M_f = const;$

2). $M_g = \max_{0\le t\le T}\left| g(t,x(t))\right| < \infty, \quad 0 < M_g = const;$

3). For all $t\in[0,T]$, $x\in X$ there exist such function $0 < L_f(t)\in C\left([0,T],R\right),$ that
$$\left| f(t,x_1) - f(t,x_2)\right| \le L_f(t)\left| x_1 - x_2\right|;$$

4). For all $t\in[0,T]$, $x\in X$ there exist such function $0 < L_g(t)\in C\left([0,T],R\right),$ that
$$\left| g(t,x_1) - g(t,x_2)\right| \le L_g(t)\left| x_1 - x_2\right|;$$

5). $\rho = M_4\left[M_f L_g(t) + M_g L_f(t)\right] < 1.$

Then the initial value problem (1), (2) has a unique solution $x(t)\in C\left([0,T],R\right),$ which can be found from the iteration

$$\begin{cases} x^0(t) = \varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right), \\ x^{m+1}(t) = J(t;x^m(t)), \ m = 0,1,2,\ldots \end{cases} \tag{24}$$

Proof. We use fixed point method and prove that the operator $J(t;x(t))$ on the right-side of the equation (23) is compressing in $C[0,T].$ For the zero approximation from the (24) we have

$$\left\| x^0(t)\right\|_{C[0,T]} \le \left|\varphi_0\right| \max_{0\le t\le T}\left| E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right)\right| \le M_3\left|\varphi_0\right|. \tag{25}$$

For arbitrary $m\in N$ we obtain that

$$\left\| x^{m+1}(t) - x^m(t)\right\|_{C[0,T]} \le$$

$$\le \max_{0\le t\le T}\int_0^t \left| K(t,s)\right| \left\| f(s,x^m(s))g(s,x^m(s)) - f(s,x^{m-1}(s))g(s,x^{m-1}(s))\right| ds \le$$

$$\le M_4\left[\left\| f(t,x^m(t))g(t,x^m(t)) - f(t,x^m(t))g(t,x^{m-1}(t))\right\|_{C[0,T]} + \right.$$

$$\left. + \left\| f(t,x^m(t))g(t,x^{m-1}(t)) - f(t,x^{m-1}(t))g(t,x^{m-1}(t))\right\|_{C[0,T]}\right] \le$$

$$\le M_4\left[\left\| f(t,x^m(t))\right\|_{C[0,T]}\left\| g(t,x^m(t)) - g(t,x^{m-1}(t))\right\|_{C[0,T]} + \right.$$

$$\left. + \left\| g(t,x^m(t))\right\|_{C[0,T]}\left\| f(t,x^m(t)) - f(t,x^{m-1}(t))\right\|_{C[0,T]}\right] \le$$

$$\le M_4\left[M_f L_g(t)\left\| x^m(t) - x^{m-1}(t)\right\|_{C[0,T]} + M_g L_f(t)\left\| x^m(t) - x^{m-1}(t)\right\|_{C[0,T]}\right] \le$$

$$\le \rho\cdot\left\| x^m(t) - x^{m-1}(t)\right\|_{C[0,T]}, \tag{26}$$

where

$$\rho = M_4\left[M_f L_g(t) + M_g L_f(t)\right].$$

From the estimates (25) and (26) implies that the operator $J(t;x(t))$ on the right-side of the equation (23) is compressing in $C[0,T]$. Hence, we deduce that the nonlinear integral equation has a unique solution in the segment $[0,T]$. So, we proved that the initial value problem (1), (2) has a unique solution in the segment $[0,T]$.

Theorem 3. Let be fulfilled the conditions of the Theorem 2. Then the solution of the problem (1), (2) is continuously dependent from the initial data (2).

Proof. Let φ_{01} and φ_{02} are two values of the initial data φ_0 such that $|\varphi_{01} - \varphi_{02}| < \delta(\varepsilon)$, where $0 < \delta$ is small number and depends from the given small number $0 < \varepsilon$. Let $x_1(t)$ and $x_2(t)$ are two solutions of the nonlinear integral equation (23), corresponding to φ_{01} and φ_{02}. We will prove that $|x_1(t) - x_2(t)| < \varepsilon$.

So, from the integral equation (23) we have

$$\|x_1(t) - x_2(t)\|_{C[0,T]} \le M_3|\varphi_{01} - \varphi_{02}| + \rho \cdot \|x_1(t) - x_2(t)\|_{C[0,T]} < M_3\delta + \rho \cdot \|x_1(t) - x_2(t)\|_{C[0,T]}.$$

From last estimate we obtain the proof of the theorem

$$\|x_1(t) - x_2(t)\|_{C[0,T]} < \frac{M_3}{1-\rho}\delta,$$

if we put that

$$\varepsilon = \frac{M_3}{1-\rho}\delta.$$

Abstract. In this article a nonlinear initial value problem (1), (2) for a Gerasimov-Caputo type fractional differential equation with degeneration is considered in the case of $0 < \alpha \le 1$. The right-hand side of the equation (1) consists product of two nonlinear functions $f(t,x(t)) \cdot g(t,x(t))$. The solution of this fractional differential equation is studied in the class of regular functions. A nonlinear integral equation (23) is obtained by using the Mittag-Leffler function. In proof of one valued solvability of the problem the method of successive approximations in combination with the method of compressing mapping is applied. The continuous dependence of solution of the problem on initial data also is studied.

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