ISNN:2181-1296

SHAROF RASHIDOV NOMIDAGI
SAMARQAND DAVLAT UNIVERSITETI

САМАРКАНДСКИЙ ГОСУДАРСТВЕННЫЙ
УНИВЕРСИТЕТ ИМЕНИ ШАРОФА РАШИДОВА

SAMARKAND STATE UNIVERSITY NAMED
AFTER SHAROF RASHIDOV

ILMIY AXBOROTNOMA

НАУЧНЫЙ ВЕСТНИК
SCIENTIFIC JOURNAL

ANIQ VA TABIIY FANLAR SERIYASI

Matematika, Mexanika, Informatika
Fizika, Kimyo, Biologiya, Geografiya

№ 3/2 (151)
2025

ISSN 2181-1296

ILMIY AXBOROTNOMA

НАУЧНЫЙ ВЕСТНИК SCIENTIFIC JOURNAL

2025-yil, 3-son (151/2) ANIQ VA TABIIY FANLAR SERIYASI

Matematika, Mexanika, Informatika, Fizika, Kimyo, Biologiya, Geografiya

Samarqand viloyat matbuot boshqarmasida ro‘yxatdan o‘tish tartibi 09-25.
Jurnal 1999-yildan chop qilina boshlagan va OAK ro‘yxatiga kiritilgan.

BOSH MUHARRIR **R. I. XALMURADOV, t.f.d., professor**
BOSH MUHARRIR O‘RINBOSARLARI: **H.A. XUSHVAQTOV, f-m.f.d., dotsent**

TAHRIRIYAT KENGASHI:

Sh.A.Alimov	- O‘zFA akademigi	**M.X.Ashurov**	- O‘zFA akademigi
S.N.Lakayev	- O‘zFA akademigi (SamDU)	**B.J.Axmedov**	- O‘zFA akademigi
M.M.Kamilov	- O‘zFA akademigi	**Q.G‘.G‘ulomov**	- O‘zFA akademigi
A.G.Yagola	- Moskva davlat universiteti professori (Rossiya)	**A.A.Zaxidov**	-Texas Nanotexnologiya instituti professori (AQSh)
Alberto Del Bimbo	-Florensiya universiteti professori (Italiya)	**N.B. Ferapontov**	- Moskva davlat universiteti professori (Rossiya)
Sh.Tugizov	-Kaliforniya universiteti professori (AQSh)	**V.L.Popov**	-Berlin texnika universiteti professori (Germaniya)
A.S.Soleev	- f.-m.f.d., professor (SamDU)	**N.F.Bunkin**	- Moskva davlat texnika universiteti professori (Rossiya)
I.A.Ikromov	- f.-m.f.d., professor (SamDU)	**Haji Akber Aisa**	-Xinjiang tibbiyot universiteti professori (Xitoy)
B.X.Xo‘Jayarov	- f.-m.f.d., professor (SamDU)	**E. A. Abduraxmonov**	- k.f.d., professor (SamDU)
L.Sabirov	-f.-m.f.d., professor (SamDU)	**N. K. Muxamadiyev**	- k.f.d., professor (SamDU)
A.Jumaboyev	-f.-m.f.d., professor (SamDU)	**L. A. Bulavin**	- Kiev milliy universiteti
X.Bozorov	-k.f.d., professor (SamDU)	**X. Q. Xaydarov**	- b.f.d., professor (SamDU)
O.Q.Quvondiqov	-f.-m.f.d. professor (SamDU)	**Z. I. Izzatullayev**	- b.f.d., professor (SamDU)
A.Z.Xasanov	-f.-m.f.d. professor (SamDU)	**A.B.Hasanov**	-tex.f.d., professor (SamDU)
Muhammad Suzuri bin Hitam	- Terengganu universiteti professori (Malayziya)	**Sh. T. Xoliqulov**	- q.-x.f.d., professor (SamDU)
K.D.Kuliyev	- f.-m.f .n. professor (SamDU)	**S. B. Abbosov**	- g.f.d., professor (SamDU)
A.M.Xolxo‘jayev	-f.-m.f.d., professor (SamDU)	**M.E. Mo‘minov**	-f.-m.f.d., professor (SamDU)
J.Abdullayev	-f.-m.f.d., professor (SamDU)	**I.I.Jumanov**	-tex.f.d., professor (SamDU)
A.R.Axatov	-tex.f.d., professor (SamDU)	**N.S.Mamatov**	-tex.f.d., professor (TIQXMMI)
D.B.Xursanov	- g.f.f.d., dotsent (SamDU)	**A.Sh.Yarmuxamedov**	-f-m.f.n., dotsent (SamDU)
F.Nazarov	-tex.f.f.d., dotsent (SamDU)	**X.S.Xaydarov**	-f-m.f.n., dotsent (SamDU)

Obuna indeksi – yakka tartibdagi obunachilar uchun **- 5583,**
tashkilot, korxonalar uchun **- 5584**

M U N D A R I J A
C O N T E N T S

№	Maqola nomi	Article title	Bet/ page
1	Kasr psevdoparabolik tenglama uchun Samarskiy-Ionkin ikki nuqtali chegaraviy shartlar **Sagdullaeva M.U., Xasanov G.A., Yuldashev T.K.**	Samarskii-Ionkin two-point boundary conditions for a fractional pseudoparabolic equation **Sagdullaeva M.U., Khasanov G.A., Yuldashev T.K.**	**5-14**
2	Gerasimov-Kaputo turidagi kasr tartibli nochiziqli differensial tenglamalar **Shukurova M.D., Xasanov G.A., Yuldashev T.K.**	Nonlinear differential equation with Gerasimov-Caputo type fractional operator and degenerator **Shukurova M.D., Khasanov G.A., Yuldashev T.K.**	**15-22**
3	NaH da lazer ta'sirida dissotsiatsiya dinamikasi **Umarov O., Jumabaev A.**	Laser-induced dissociation dynamics in NaH **Umarov O., Jumabaev A.**	**23-30**
4	Radon transformasidan foydalanish, tasvirni tanishda aylana sfera proyeksiyasini yaratish **Khatamov O., Mamanova B.**	Using the radon transform to create a projection of a circular sphere in image recognition **Khatamov O., Mamanova B.**	**31-38**
5	Flavonoid komplekslarining metal ionlari bilan sintezi, spektroskopik tadqiqotlar va supramolekulyar tizimlar **Ergashova Sh., Muxamadiyev N.**	Synthesis of flavonoid complexes with metal ions, spectroscopic studies and supramolecular systems **Ergashova Sh., Mukhamadiev N.**	**39-45**
6	2-benzil-6,7-dimetoksi-1,2,3,4-tetragidroizoxinolin hosilalari sintez reaksiyalarini o'rganish **Suyunova S.N., Quljonov J.X., Xudoyberdiyeva A.A., Urunbayeva Z.E. Z.E., Saidov A.Sh., Vinogradova V.I.**	Study of the reactions of synthesis of 2-benzyl-6,7-dimethoxy-1,2,3,4-tetrahydroisochinoline derivatives **Suyunova S.N., Kuljonov J.X., Khudoyberdieva A.A., Urunbaeva Z.E. Z.E., Saidov A.Sh., Vinogradova V.I.**	**46-50**
7	Keyingi qo'shni tugunlargacha o'zaro ta'sirlanuvchi ikki bozonli panjara gamiltoniani **Axmadova M., Axmatova Sh.**	A two-boson lattice hamiltonian with interactions up to next-neighboring sites **Akhmadova M., Akhmatova Sh.**	**51-58**
8	Sekin aylanishli magnitlangan neytron yulduzlarning plazma magnitosferasi o'zgartirilgan gravitatsiyada **Sayfiyev Sh.**	Plasma magnetosphere of slowly rotating magnetized neutron stars in modified gravity **Sayfiev Sh.**	**59-65**
9	Vaqtli qatorlar va mashinaviy o'qitish asosida sotuvdagi kechikishlarni aniqlash usullari **Nazarov F., Sayidqulov A.**	Methods for detecting delays in sales based on time series and machine learning **Nazarov F., Sayidkulov A.**	**66-71**
10	Integrallashgan matematik modellar va ularning qo'llanilishi **Safarov R., Xudoyberdiyev S.I.**	Integrated mathematical models and their applications **Safarov R., Khudoyberdiyev S.I.**	**72-77**
11	Chekli zichlikka va qator ko'rinishdagi moslangan manbaga ega Hirota tenglamasi uchun boshlang'ich qiymat masalasi **Eshbekov R.**	The initial value problem for the Hirota equation with finite density and self-consistent series source **Eshbekov R.**	**78-84**
12	L-sistein bilan modifikatsiyalangan mezog'ovak silikagel yordamida og'ir metall kationlarini ajratish **Ulugboyeva G.O., Kuliboyeva N.Sh., Uzokov J.R., Fazliyeva N.T., Muxamadiev N.Q.**	Removal of heavy metal cations using mesoporous silicagel modified with L-cysteine **Ulugboeva G.O., Kuliboeva N.Sh., Uzokov J.R., Fazlieva N.T., Mukhamadiev N.Q.**	**85-90**
13	Titan nano oksidi sintezi va uning fotokatalitik xossalarini zarracha o'lchamiga bog'liqligi **Mamaziyeva S., Avazova N., Tashmatova R., Muxamadiev N., Muxamadiev A., Uzokov J.**	Synthesis of nano titanium oxide and dependence of its photocatalytic properties on particle size **Mamazieva S., Avazova N., Tashmatova R., Mukhamadiev N., Mukhamadiev A., Uzokov J.**	**91-96**
14	$SrTiO_3$ nanozarrachalarini sintez qilish va ularning o'lcham-fotokatalitik xossalari **Hayitova M., Baxranova M.A., Muxamadiyev N.Q.**	Synthesis of $SrTiO_3$ and its size-dependent photocatalytic properties **Hayitova M., Bakhranova M.A., Mukhamadiyev N.Q.**	**97-102**

№	Maqola nomi	Article title	Bet/ page
15	Rekreatsiya maqsadlarida tabiatdan foydalanishni boshqarish **Rafikova N.**	Management of nature utilization for recreational purposes **Rafikova N.**	**103-110**
16	Modifikatsiyalangan faollashgan ko‘mir yordamida eritma tarkibidagi organik bo‘yoqlarni ajratish **Ergasheva D., Muxamadiyev A., Muxamadiyev N., Uzokov J.**	Separation of organic dyes from solutions using modified activated carbon **Ergasheva D., Mukhamadiyev A., Mukhamadiyev N., Uzokov J.**	**111-116**
17	Neyron tarmoqli modellar asosida qandli diabetni tashxislashning intellektual algoritmlari **Sariyev Sh.N.**	Intelligent algorithms for diagnosing diabetes mellitus based on neural network models **Sariev Sh.N.**	**117-123**
18	Mahalliy kaolindan seolitlar sintez va fizik-kimyoviy tahlili **Magdiev Sh., Majidova F., Nasimov A., Tashpulatov X., Toshpulatov D., Mirzayev Sh., Norboev K., Normuradov O.**	Synthesis and physicochemical analysis of zeolite from local kaolin **Magdiev Sh., Majidova F., Nasimov A., Tashpulatov Kh., Toshpulatov D., Mirzaev Sh., Norboev K., Normuradov O.**	**124-129**
19	Densenet modelini transfer o‘qitish asosida masofadan zondlash sahnalarini tasniflash aniqligini oshirish maqolaning o‘zbek tilidagi nomi **Yusupov O.R., Eshonqulov E.Sh.**	Enhancing the accuracy of remote sensing scene classification using transfer learning with densenet **Yusupov O.R., Eshonkulov E.Sh.**	**130-144**
20	Neft va gaz quduqlari minerallashgan qatlam suvlarining kimyoviy tarkibi va fizik-kimyoviy xususiyatlarini o‘rganish **Kenjaev N.A., Igamkulova N.A.**	Chemical composition and physico-chemical properties of mineralized formation waters from oil and gas wells **Kenjaev N.A., Igamkulova N.A.**	**145-148**

UDC: 517.984 **DOI:https://doi.org/10.59251/2181-3973.2025.v1.138.1.3440**

A TWO-BOSON LATTICE HAMILTONIAN WITH INTERACTIONS UP TO NEXT-NEIGHBORING SITES

M.Akhmadova, Sh.Akhmatova
Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.

M.Akhmadova Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.

Received: 16.06.2025
Revised: 30.06.2025
Accepted: 13.07.2025
Published: 31.07.2025

Publisher: Samarkand State University named after Sharof Rashidov.

Abstract. We analyze the spectral properties of a family of discrete Schrödinger operators $H_\mu(K)$, $K \in T^2$. We partition the real line into five connected components, within which the number of eigenvalues of the operator $H_\mu(0)$ lying below and above the essential spectrum is constant. Necessary and sufficient conditions on μ are established for $H_\mu(0)$ to have exactly α eigenvalues below or above the essential spectrum, where $\alpha \in \{0,1,2,3,4\}$. Finally, we provide sharp lower bound for the number of eigenvalues of the operator $H_\mu(K)$, $K \in T^2$, lying outside the essential spectrum.

Keywords: Discrete Schrödinger operator, eigenvalue, essential spectrum, lattice, two-particle system, virtual level.

Keyingi qo‘shni tugunlargacha o‘zaro ta’sirlanuvchi ikki bozonli panjara gamiltoniani

Annotatsiya. Biz panjaradagi diskret Schryodinger operatorlari oilasi $H_\mu(K)$ $K \in T^2$ning spektral xossalarini tahlil qilamiz. Haqiqiy o‘qlarni beshta bog‘langan sohalarga ajratamiz, bunda operator $H_\mu(0)$ ning muhim spektrdan pastda yoki yuqorida joylashgan xos qiymatlari soni har bir sohada o‘zgarmas bo‘ladi. $H_\mu(0)$ operatorning muhim spektrdan quyida yoki yuqorida aniq α ta, bunda $\alpha \in \{0,1,2,3,4\}$, xos qiymatga ega bo‘lishi uchun zarur va yetarli shartlar topiladi. Shuningdek, $K \in T^2$ uchun $H_\mu(K)$ operatorining muhim spektrdan tashqarida joylashgan xos qiymatlari soni uchun aniq quyi baho topamiz.

Kalit so‘zlar: Diskret Shreder operatori, xos qiymat, muhim spektr, panjara (to‘r), ikki zarrali sistema, virtual sath.

Гамильтониан двух бозонов на решетке с взаимодействиями до следующих соседних узлов

Аннотация. Мы анализируем спектральные свойства семейства дискретных операторов Шрёдингера $H_\mu(K)$, $K \in T^2$. Делим действительные оси на пять связных компонентов, в пределах каждой из которых количество собственных значений оператора $H_\mu(0)$ лежащих ниже или выше основного спектра, остается постоянным. Установлены необходимые и достаточные условия на параметр μ, при которых оператор $H_\mu(0)$ имеет ровно $\alpha \in \{0,1,2,3,4\}$ собственных значений ниже или выше основного спектра. В заключение приводится точная нижняя оценка числа собственных значений оператора $H_\mu(K)$, $K \in T^2$, лежащих вне основного спектра.

Ключевые слова: дискретный оператор Шрёдингера, собственное значение, существенный спектр, решётка, двухчастичная система, виртуальный уровень.

Introduction. Lattice-based models play a significant role across many areas of physics. A particular class of interest includes lattice Hamiltonians involving a small, fixed number of particles [31], which can be seen as simplified analogs of the Bose- or Fermi-Hubbard models. These few-particle Hamiltonians are already known to possess considerable theoretical value on their own [2, 4, 5, 18, 26]. Moreover, such discrete Hamiltonians can serve as effective approximations to their continuous equivalents [9], enabling the investigation of few-particle phenomena within the framework of bounded operator theory.

Unlike in the continuous setting, the separation of center-of-mass motion is not possible in few-particle lattice systems. Nevertheless, the system's discrete translational symmetry makes it possible to apply the Floquet–Bloch decomposition (see, for instance, [3, Sec. 4]).

Particularly, the total n-particle lattice Hamiltonian H in the (quasi)momentum space representation can be decomposed as the von Neumann direct integral

$$H \simeq \int_{K\in T^d}^{\oplus} H(K)dK,$$

where T^d is the d-dimensional torus. The fiber Hamiltonian $H(K)$ acting in the respective functional Hilbert space on $T^{(n-1)d}$ is non trivially dependent on the quasimomentum $K \in T^d$ (see e.g., [3,31]).

In the present study, we investigate the spectral characteristics of the fiber Hamiltonians $H(K)$,, where $K \in T^2$, operating in the Hilbert space $L^{2,e}(T^2)$. These operators are defined as

$$H_\mu(K) := H_0(K) + V_\mu,$$

where $H_0(K)$ represents the unperturbed part and V_μ denotes the interaction term.

$$(H_0(K)f)(p) = E_K(p)f(p),$$

with

$$E_K(p) := 4\sum_{i=1}^{2}(1 - cos\frac{K_i}{2}cosp_i)$$

and the term V_μ represents the total interaction potential. The parameter μ characterizes the strength of interactions between particles situated on the same site, as well as those on adjacent and next-nearest neighboring sites of the lattice.

Notice that V_μ does not depend on K at all. Surely, the operators $H_0(K)$ and V_μ, $\mu \in R$ are both bounded and self-adjoint. Since V_μ is finite rank, the essential spectrum of $H_{\gamma\lambda\mu}(K)$ coincides with the spectrum of $H_0(K)$, i.e. $[E_{min}(K), E_{max}(K)]$ where

$$E_{min}(K) := 4\sum_{i=1}^{2}(1 - cos\frac{K_i}{2}), E_{max}(K) := 4\sum_{i=1}^{2}(1 + cos\frac{K_i}{2}).$$

The primary goal of this study is to determine precisely both the number and the position of the eigenvalues of the operator $H_\mu(0)$ that lie outside its essential spectrum. This result is then used to derive sharp bounds on the number of eigenvalues of $H_\mu(K)$ for all $K \in T^2$

In order to carry out this analysis, we focus on the Fredholm determinant $\Delta_\mu(z)$, which is associated with the discrete Schrödinger operator $H_\mu(0)$. It is a well-known result that the eigenvalues of $H_\mu(0)$ correspond exactly to the zeros of $\Delta_\mu(z)$ (see [23]).

References [23, 25] examine Schrödinger operators defined on the lattice Z^d, for $d = 1,2$, corresponding to a system of two bosons subject to zero-range interaction at a single site (with $\lambda \in R$) and an additional interaction between particles located on nearest-neighbor sites (with $\mu \in R$).

The operator $H_\mu(0)$ may have zero, one, or two eigenvalues, which can lie either below or above the essential spectrum.

The discret schrödinger operator and essential spectrum. The fiber operator $H_\mu(K)$ for $K \in T^2$, a self-adjoint operator on $L^{2,e}(T^2)$, is given by

$$H_\mu(K) := H_0(K) + V_\mu,$$

where $H_0(K)$ is multiplication by

$$E_K(p) := 4\sum_{i=1}^{2}\left(1 - cos\frac{K_i}{2}cosp_i\right),$$

and V_μ is the perturbation defined as

$$V_\mu f)(p) = \frac{\mu}{2\pi^2}\sum_{i=1}^{2} cos2p_i \int_{T^2} cos2q_i f(q)dq \quad (1)$$
$$\frac{+\mu}{\pi^2} cosp_1 cosp_2 \int_{T^2} cosq_1 cosq_2 f(q)dq$$
$$\frac{+\mu}{\pi^2} sinp_1 sinp_2 \int_{T^2} sinq_1 sinq_2 f(q)dq.$$

We note that, K represents the two-particle quasimomentum. Since the rank of V_μ (dependent on μ) is at most four, Weyl's theorem implies that the essential spectrum $\sigma_{ess}(H_\mu(K))$ coincides with the spectrum of $H_0(K)$ for all $K \in T^2$:

$$\sigma_{ess}(H_\mu(K)) = \sigma(H_0(K)) = [E_{min}(K), E_{max}(K)], \quad (2)$$

where

$$E_{min}(K) := \min_{p\in T^2} E_K(p) = 2\sum_{i=1}^{2} (1 - cos\frac{K_i}{2}) \geq E_{min}(0) = 0,$$

$$E_{max}(K) := \max_{p\in T^2} E_K(p) = 2\sum_{i=1}^{2} (1 + cos\frac{K_i}{2}) \leq E_{max}(0) = 16.$$

Theorem 1 *Let $\mu \in$. Then the following assertions hold:*

i) If the operator $H_\mu(0)$ has α eigenvalues lying below the essential spectrum then, for any $K \in T^2$, the fiber Hamiltonian $H_\mu(K)$ has at least α eigenvalues lying below the essential spectrum .

ii) If theoperator $H_\mu(0)$ has β eigenvalues lying above the essential spectrum then, for any $K \in T^2$, the fiber Hamiltonian $H_\mu(K)$ has at least β lying eigenvalues above the essential spectrum .

It is worth recalling that the central objective of this paper is to analyze both the quantity and the position of the eigenvalues of the operator H_μ (0) that lie outside its essential spectrum. Building on this result, we then determine the number of eigenvalues of the operator H_μ (K), for all K∈T^2.

The Lippmann–Schwinger operator. Let us consider the following system of orthonormal vectors in $L^{2,e}(T^2)$:

$$\alpha_1(p) = \frac{1}{\pi} cos2p_1, \alpha_2(p) = \frac{1}{\pi} cos2p_2, \quad (3)$$
$$\alpha_3(p) = \frac{1}{\pi} cosp_1 cosp_2, \alpha_4(p) = \frac{1}{\pi} sinp_1 sinp_2.$$

Utilizing equation (1), the potential V_μ can be expressed in terms of the orthonormal system (3) as follows:

$$V_\mu f = \mu\sum_{i=1}^{4} (f, \alpha_i)\alpha_i, \quad (4)$$

where $(\cdot,\cdot)$ is the inner product in $L^{2,e}(T^2)$. For any $z \in C[0,16$ we define (the transpose of) the Lippmann-Schwinger operator (see., e.g., [30]) as

$$B_\mu(0,z) = -V_\mu R_0(0,z), \quad (5)$$

where $R_0(0,z) := [H_0(0) - zI \quad ^{-1}, z \in C[0,16]$, is the resolvent of the operator $H_0(0)$.

Lemma 1 *For every $\mu \in R$ the number $z \in C[0,16$ is an eigenvalue of $H_\mu(0)$ if and only if the number 1 is an eigenvalue for $B_\mu(0,z)$.*

Proof. Let $\mu \in R$ and $z \in C[0,16$. Then, the resolvent operator $R_0(0,z) =$ is well-defined, where I is the identity operator on $L^{2,e}(T^2)$. It shows that the equation

$$H_\mu(0)\varphi = z\varphi, \varphi \in L^{2,e}(T^2)$$

rewritten as

$$(H_0(0) - zI)\varphi = -V_\mu\varphi. \quad (6)$$

By multiplying the equation (6) by $R_0(0,z) :=$ we obtain that

$$\varphi = -V_\mu R_0(0,z)\varphi, \varphi \in L^{2,e}(T^2). \quad (7)$$

Analogously, by multiplying the equation (7) by $(H_0(0) - zI)$, we obtain equation (6). This equivalence between (6) and (7) proves the assertion.

The representation (4) yields the equivalence of the Lippmann-Schwinger equation

$$B_\mu(0,z)\varphi = \varphi, \varphi \in L^{2,e}(T^2)$$

to the following algebraic linear system in the variables $x_i := (\varphi, \alpha_i), i = 1,2,3,4$:

$$\begin{cases} [1+\mu a_{11}(z)]x_1 + \mu a_{12}(z)x_2 + \mu a_{13}(z)x_3 + \mu a_{14}(z)x_4 = 0, \\ \mu a_{12}(z)x_1 + [1+\mu a_{22}(z)]x_2 + \mu a_{23}(z)x_3 + \mu a_{24}(z)x_4 = 0, \\ \mu a_{13}(z)x_1 + \mu a_{23}(z)x_2 + [1+\mu a_{33}(z)]x_3 + \mu a_{34}(z)x_4 = 0, \\ \mu a_{14}(z)x_1 + \mu a_{24}(z)x_2 + \mu a_{34}(z)x_3 + [1+\mu a_{44}(z)]x_4 = 0 \end{cases} \quad (8)$$

where

$$a_{ij}(z) := \int_{T^2} \frac{\alpha_i(p)\alpha_j(p)dp}{E_0(p)-z}, i,j = 1,2,3,4. \quad (9)$$

Let

$$\Delta_\mu(z) := det\left[I - B_\mu(0,z)\right] \quad (10)$$

$$\begin{vmatrix} 1+\mu a_{11}(z) & \mu a_{12}(z) & \mu a_{13}(z) & \mu a_{14} \\ \mu a_{12}(z) & 1+\mu a_{22}(z) & \mu a_{23}(z) & \mu a_{24}(z) \\ \mu a_{13}(z) & \mu a_{23}(z) & 1+\mu a_{33}(z) & \mu a_{34}(z) \\ \mu a_{14}(z) & \mu a_{24}(z) & \mu a_{34}(z) & 1+\mu a_{44}(z) \end{vmatrix},$$

where $z \in C[0,16$.

Lemma 2 *The following conditions hold for the functions $a_{ij}(\cdot)$, $i,j \in \{1,2,3,4\}$ on $R[0,16$:*

i) $a_{ji} = a_{ij}, \forall i,j \in \{1,2,3,4\}$.

ii) $a_{14} = a_{24} = a_{34} \equiv 0$,

iii) (iii) $a_{11} = a_{22}, \wedge a_{13} = a_{23}$.

Proof. (i) The definition (9) directly yields the symmetry $a_{ij} = a_{ji}$ for all $i,j \in \{1,2,3,4\}$.

(ii)The oddness of the corresponding integrable function, combined with a property of the Lebesgue integral, implies that $a_{14}(z) = a_{24}(z) = a_{34}(z) = 0$, for all $z \in C[0,16$.

(iii) The definitions (9) and (3) combined with direct calculations gives the result.

The following lemma describes the relations between the operator $H_\mu(0)$ and the function $\Delta_\mu()$.

Lemma 3 *For a given $\mu \in R$ and $z \in R[0,16$, z is an eigenvalue of $H_\mu(0)$ with multiplicity m if and only if z is a zero of $\Delta_\mu(\cdot)$ with the same multiplicity m. Furthermore, $\Delta_\mu(\cdot)$ has at most four zeros in $R[0,16$.*

The proof of this lemma follows a standard argument (see, for instance, [23, 22]).

Lemma 4 *The Fredholm determinant $\Delta_\mu(z)$ can be represented as follow:*

$$\Delta_\mu(z) = \Delta_\mu^{(1)}(z) \cdot \Delta_\mu^{(2)}(z) \cdot \Delta_\mu^{(3)}(z), \quad (11)$$

where

$$\Delta_\mu^{(1)}(z) = 1 + \mu(a_{11}(z) - a_{12}(z)), \quad (12)$$

$$\Delta_\mu^{(2)}(z) = \begin{vmatrix} 1+\mu(a_{11}(z)+a_{12}(z)) & \sqrt{2}\mu a_{13}(z) \\ \sqrt{2}\mu a_{13}(z) & 1+\mu a_{33}(z) \end{vmatrix}, \quad (13)$$

$$\Delta_\mu^{(3)}(z) = 1 + \mu a_{44}(z). \quad (14)$$

Lemma 2.4 proved using Lemma 2.2 and simple calculations.

Proposition 1 *The functions $a_{ij}(z)(i,j = 1,2)$ are real-valued for $z \in R[0,16$, strictly increasing and positive in $(-\infty, 0)$, strictly increasing and negative in $(16, \infty)$. Furthermore, the following asymptotics are valid:*

$$a_{11}(z) = \begin{cases} \frac{-1}{2\pi} ln(-z) + \frac{15ln2 - 60\pi + 176}{6\pi} + o(1), asz \nearrow 0 \\ \frac{1}{2\pi} ln(z-16) - \frac{15ln2 - 60\pi + 176}{6\pi} + o(1), asz \searrow 16 \end{cases},$$

$$a_{12}(z)=\begin{cases}\dfrac{-1}{2\pi}ln(-z)+\dfrac{15ln2-16}{6\pi}+o(1), asz\nearrow 0,\\ \dfrac{1}{2\pi}ln(z-16)-\dfrac{15ln2-16}{6\pi}+o(1), asz\searrow 16,\end{cases}$$

$$a_{13}(z)=\begin{cases}\dfrac{-1}{2\pi}ln(-z)+\dfrac{15ln2+12\pi-52}{6\pi}+o(1), asz\nearrow 0,\\ \dfrac{1}{2\pi}ln(z-16)-\dfrac{15ln2+12\pi-52}{6\pi}+o(1), asz\searrow 16,\end{cases}$$

$$a_{33}(z)=\begin{cases}\dfrac{-1}{2\pi}ln(-z)+\dfrac{15ln2-6\pi+8}{6\pi}+o(1), asz\nearrow 0,\\ \dfrac{1}{2\pi}ln(z-16)-\dfrac{15ln2-6\pi+8}{6\pi}+o(1), asz\searrow 16,\end{cases}$$

$$a_{44}(z)=\begin{cases}\dfrac{3\pi-8}{3\pi}+o(1), asz\nearrow 0,\\ \dfrac{-3\pi-8}{3\pi}+o(1), asz\searrow 16.\end{cases}$$

where $ln(-z)$ and $ln(z-16)$ denote those branches of the corresponding analytic functions that are real for $-z>0$ and $z>16$, respectively.

Proposition 1 can be proved as Proposition 4.2 in [23].

Lemma 5 *The following assertions are true:*

(i) $\lim\limits_{z\to\pm\infty}\Delta_{\mu}^{(i)}(z)=1, i\in\{1,2,3\}$,

(ii) *The function* $\Delta_{\mu}^{(1)}(z)$ *is real valued for* $z\in R[0,16$ *and possesses the following asymptotics:*

$$\Delta_{\mu}^{(1)}(z)=\begin{cases}1+\mu\dfrac{32-10\pi}{\pi}+o(1), z\nearrow 0;\\ 1-\mu\dfrac{32-10\pi}{\pi}+o(1), z\searrow 16,\end{cases}$$

(iii) *The function* $\Delta_{\mu}^{(2)}(z)$ *is real valued for* $z\in R[0,16$ *and possesses the following asymptotics:*

$$\Delta_{\mu}^{(2)}(z)=$$

where $Q_1^{\pm}(\mu)=\mu(\mu\mp\frac{9\pi}{4(32+\pi)})$.

(iv) *The function* $\Delta_{\mu}^{(3)}(z)$ *is real valued for* $z\in R[0,16$ *and possesses the following asymptotics:*

$$\Delta_{\mu}^{(1)}(z)=\begin{cases}1+\dfrac{3\pi-8}{3\pi}, z\nearrow 0;\\ 1-\dfrac{3\pi-8}{3\pi}, z\searrow 16,\end{cases}$$

To articulate our results, we introduce the following essential numbers:

$$\begin{aligned}&\mu_1^{\pm}=\pm\tfrac{\pi}{2(16-5\pi)}\approx\pm 0.1859\ldots\\ &\mu_2^{\pm}=\pm\tfrac{9\pi}{4(32+\pi)}\approx\pm 0.2156\ldots\\ &\mu_3^{\pm}=\pm\tfrac{3\pi}{3\pi-8}\approx\pm 6.6149\ldots.\end{aligned}\qquad(15)$$

The following directly and appropriately follows from (15):

$$\mu_1^{-<\mu_2^{-<\mu_3^{-<0<\mu_3^{+<\mu_2^{+<\mu_1^{+}}}}}}\qquad(16)$$

Lemma 6 *The following assertions hold:*

i) *if* $\mu<\mu_1^-$, $\Delta_{\mu}^{(1)}(z)$ *has exactly one zero in* $(-\infty,0$,

ii) *if* $\mu_1^{-<\mu<\mu_1^+}$, $\Delta_{\mu}^{(1)}(\mu)$ *has no zero in* $R[0,16$,

iii) *if* $\mu > \mu_1^+$, $\Delta_\mu^{(1)}(\mu)$ *has exactly one zero in* $(16, +\infty)$.

Proof. (i) Let us assume that $\mu < \mu_1^-$ and define $\Delta^{(1)}(z) := a_{11}(z) - a_{12}(z)$. According to definition (9), the function $\Delta^{(1)}(z)$ is given by the integral:

$$\Delta^{(1)}(z) = \frac{1}{\pi^2}\int_{T^2} \frac{cos2p_1(cos2p_1 - cos2p_2)}{\varepsilon_0(p) - z}dp = \frac{1}{4\pi^2}\int_{T^2} \frac{(cos2p_1 - cos2p_2)^2}{\varepsilon_0(p) - z}dp.$$

The properties of this integral imply that $\Delta^{(1)}(z)$ is positive and strictly increasing in $(-\infty, 0)$ and negative and strictly increasing in $(16, +\infty)$. It follows that the function $\Delta_\mu^{(1)}(z) = 1 + \mu\Delta^{(1)}(z)$ is strictly decreasing for $\mu < \mu_1^{-<0}$.

From Lemma 5, (i)-(ii), we can deduce that

$$\lim_{z\to-\infty} \Delta_\mu^{(1)}(z) = 1, \lim_{z\nearrow 0} \Delta_\mu^{(1)}(z) = 1 + \frac{2(16-5\pi)}{\pi}\mu < 0 \quad (17)$$

and

$$\lim_{z\to+\infty} \Delta_\mu^{(1)}(z) = 1, \lim_{z\searrow 16} \Delta_\mu^{(1)}(z) = 1 - \frac{2(16-5\pi)}{\pi}\mu > 0. \quad (18)$$

The behavior of $\Delta_\mu^{(1)}(z)$ as $z \to -\infty$ and $z \nearrow 0$, as described by equations (17), combined with its strict monotonicity and continuity, implies that it possesses exactly one zero in the interval $(-\infty, 0)$. Similarly, the limits in (18) demonstrate that $\Delta_\mu^{(1)}(z)$ has no zeros in $(16, +\infty)$.

Note that, the proofs of Lemma 6 items (ii) and (iii) are analogous to that of (i).

Lemma 7 *The following statements are true:*

i) *if* $\mu < \mu_3^-$, $\Delta_\mu^{(3)}(z)$ *has exactly one zero in* $(-\infty, 0)$,

ii) *if* $\mu_2^{-<\mu<\mu_2^+}$, $\Delta_\mu^{(3)}(z)$ *has no zeros in* $R[0,16$,

iii) *if* $\mu > \mu_2^+$,$\Delta_\mu^{(3)}(z)$ *has exactly one zero in* $(16, +\infty)$.

We will now investigate the location and number of zeros of the function $\Delta_\mu^{(3)}(z)$ depending on the parameter μ.

Lemma 8 *The following assertions hold:*

i) *if* $\mu < \mu_3^-$, $\Delta_\mu^{(2)}(z)$ *has exactly two zeros in* $(-\infty, 0)$.

ii) *if* $\mu_3^{-<\mu<0}$, $\Delta_\mu^{(2)}(z)$ *has exactly one zero in* $(-\infty, 0)$.

iii) *if* $0 < \mu < \mu_3^+$, $\Delta_\mu^{(2)}(z)$ *has exactly one zero in* $(16, +\infty)$.

iv) *if* $\mu > \mu_3^+$ *then* $\Delta_\mu^{(2)}(z)$ *has exactly two zeros in* $(16, +\infty)$.

Proof. (i) Assume that $\mu < \mu_3^-$. Due to definition (9), we have the following integral expression:

$$a_{11}(z) + a_{12}(z) = \frac{1}{\pi^2}\int_{T^2} \frac{cos2p_1(cos2p_1+cos2p_2)}{\varepsilon_0(p)-z}dp = \frac{1}{2\pi^2}\int_{T^2} \frac{(cos2p_1+cos2p_2)^2}{\varepsilon_0(p)-z}dp. \quad (19)$$

The properties of the integral demonstrate that $a_{11}(z) + a_{12}(z)$ is positive and strictly increasing for $z \in (-\infty, 0)$ and negative and strictly increasing for $z \in (16, -\infty)$.

The function $\delta(z) := 1 + \mu(a_{11}(z) + a_{12}(z))$ is continuous and strictly decreasing in $(-\infty, 0)$.

The explicit definition of $a_{11}(z) + a_{12}(z)$ in (9) shows that $\delta(z)$ approaches 1 as $z \to -\infty$ and $-\infty$ as $z \nearrow 0$:

$$\lim_{z\to-\infty} \delta(z) = 1, \lim_{zZ0} \delta(z) = -\infty. \quad (20)$$

As a consequence of (20) and the properties of $\delta(z)$ (strict monotonicity and continuity), the function $\delta(z)$ has a unique zero, z_{11}, within $(-\infty, 0)$. Since $\delta(z_{11}) = 1 + \mu(a_{11}(z_{11}) + a_{12}(z_{11})) = 0$, it follows that

$$\Delta_\mu^{(2)}(z_{11}) = (1 + \mu(a_{11}(z_{11}) + a_{12}(z_{11})))(1 + \mu a_{33}(z_{11})) - 2\mu^2 a_{13}^2(z_{11}) = \quad (21)$$

$$-2\mu^2 a_{13}^2(z_{11}) < 0. \quad (22)$$

It is known that, for $\mu < \mu_3^{-<0}$

$$Q_{\mu}^{-=\mu(\mu+\frac{9\pi}{4(128+\pi)})>0} \tag{23}$$

The inequality (21) and Lemma 2.2 (i) yield that

$$\lim_{z\nearrow 0} \Delta_{\mu}^{(2)}(z) = +\infty. \tag{24}$$

Then Lemma 5, relations (21) and (24) yield that

$$\lim_{z\to\pm\infty} \Delta_{\mu}^{(1)}(z) = 1, \Delta_{\mu}^{(2)}(z_{11}) < 0, \lim_{zZ0} \Delta_{\mu}^{(2)}(z) = +\infty.$$

From this, the existence of real numbers z_{21} and z_{22} satisfying the subsequent condition is evident

$$z_{21} < z_{11} < z_{22} < 0 \tag{25}$$

and

$$\Delta_{\mu}^{(2)}(z_{21}) = \Delta_{\mu}^{(3)}(z_{22}) = 0 \tag{26}$$

Since Lemma 3 the operator $H_{\mu}(0)$ has at most two eigenvalues in $R[0,16$. Therefore, it has no eigenvalues in $(16,+\infty)$ when $\mu < \frac{-9\pi}{4(32+\pi)}$.

We now prove (ii). Assume that $\frac{-9\pi}{4(32+\pi)} \le \mu < 0$. Then

$$Q_1^{-(\mu)=\mu\left[\mu+\frac{9\pi}{4(32+\pi)}\right]<0} \tag{27}$$

and

$$Q_1^{+(\mu)=\mu\left[\mu-\frac{9\pi}{4(32+\pi)}\right]>0.} \tag{28}$$

The inequality (27) and Lemma 5 yield that

$$\lim_{z\to-\infty} \Delta_{\mu}^{(2)}(z) = 1 \wedge \lim_{zZ0} \Delta_{\mu}^{(2)}(z) = -\infty. \tag{29}$$

Similarly, the inequality (28) and Lemma 2.6 obey that

$$\lim_{z\to+\infty} \Delta_{\mu}^{(2)}(z) = 1 \wedge \lim_{z} 16\,\Delta_{\mu}^{(2)}(z) = +\infty. \tag{30}$$

The change in sign of the continuous function $\Delta_{\mu}^{(2)}(z)$ at the endpoints of $(-\infty,0)$, as shown by (29), guarantees at least one zero in this interval. The possibility of two or more zeros, implying at least three zeros due to the sign change, is ruled out by Lemma 3, which contradicts the two zeros of $H_{\mu}(0)$ established by Lemmas 6 and 7. Thus, $\Delta_{\mu}^{(2)}(z)$ has exactly one zero in $(-\infty,0)$.

Similarly, the consistent sign of $\Delta_{\mu}^{(2)}(z)$ at the boundaries of $(16,+\infty)$, according to (30), means any zero in this interval would require at least two zeros there. This would result in at least three zeros outside the essential spectrum, a contradiction that proves $\Delta_{\mu}^{(2)}(z)$ has no zeros in $(16,+\infty)$.

Note that, the assertions (iii) and (iv) can be proved in a similar way as (ii) and (i), respectively.

The following lemma describes the location and number of the operator $H_{\mu}(0)$ depending on the parameter $\mu \in R$.

Lemma 9 *Let $\mu \in R$. Then, the following assertions are demonstrated:*

i) if $\mu < \mu_1^-$ resp. $\mu > \mu_1^+$ then the operator $H_{\mu}(0)$ has exactly four eigenvalues outside of the essential spectrum and they belong to $(-\infty,0)$ resp. $(16,+\infty)$.

ii) if $\mu_1^{-<\mu<\mu_2^-}$ resp. $\mu_2^{+<\mu<\mu_1^+}$ then the operator $H_{\mu}(0)$ has exactly three eigenvalues outside of the essential spectrum and they belong to $(-\infty,0)$ resp. $(16,+\infty)$.

iii) if $\mu_2^{-<\mu<\mu_3^-}$ resp. $\mu_3^{+<\mu<\mu_2^+}$ then the operator $H_{\mu}(0)$ has exactly two eigenvalues outside of the essential spectrum and they belong to $(-\infty,0)$ resp. $(16,+\infty)$.

iv) if $\mu_3^{-<\mu<0}$ resp. $0 < \mu < \mu_3^+$ then the operator $H_{\mu}(0)$ has exactly one eigenvalue outside of the essential spectrum and it belongs to $(-\infty,0)$ resp. $(16,+\infty)$.

Lemma 9 is proven directly using the results of Lemma 6, Lemma 7, Lemma 8 and Lemma 3.

Next theorem states a sharp estimates for the number of eigenvalues of the operator $H_\mu(K)$, over all $K \in T^2$depending only parameter $\mu \in R$.

Lemma 10 *Let $K \in T^2$ and $\mu \in R$. Then the following statements hold:*

(i) if $\mu < \mu_1^-$ resp. $\mu > \mu_1^+$ then the operator $H_\mu(K)$ has at least four eigenvalues outside of the essential spectrum and they belong to $(-\infty, 0)$ resp. $(16, +\infty)$.

(ii) if $\mu_1^- < \mu < \mu_2^-$ resp. $\mu_2^+ < \mu < \mu_1^+$ then the operator $H_\mu(K)$ has at least three eigenvalues outside of the essential spectrum and they belong to $(-\infty, 0)$ resp. $(16, +\infty)$.

(iii) if $\mu_2^- < \mu < \mu_3^-$ resp. $\mu_3^+ < \mu < \mu_2^+$ then the operator $H_\mu(K)$ has at least two eigenvalues outside of the essential spectrum and they belong to $(-\infty, 0)$ resp. $(16, +\infty)$.

(iv) if $\mu_3^- < \mu < 0$ resp. $0 < \mu < \mu_3^+$ then the operator $H_\mu(K)$ has at least one eigenvalue outside of the essential spectrum and it belongs to $(-\infty, 0)$ resp. $(16, +\infty)$.

Lemma 10 is proven directly using the results of Lemma 10 and Teorema 1.

References

1. Albeverio S., Lakaev S. N., Muminov Z. I. Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics, Ann. Henri Poincaré. 5 (2004), 743-772.
2. Albeverio S., Lakaev S. N., Makarov K. A., Muminov Z. I. The Threshold Effects for the Two-particle Hamiltonians on Lattices, Comm. Math. Phys. http://dx.doi.org/10.1007/s00220-005-1454-y 262 (2006), 91–115.
3. Albeverio S., Lakaev S. N., Khalkhujaev A. M. Number of Eigenvalues of the Three-Particle Schrodinger Operators on Lattices, Markov Process. Relat. Fields. 18 (2012), 387-420.
4. Bach V., W. de Siqueira Pedra, Lakaev S.N. Bounds on the discrete spectrum of lattice Schrödinger operators. J. Math. Phys. 59:2 (2017), 022109.
5. Faddeev L.D., Merkuriev S.P. Quantum Scattering Theory for Several Particle Systems (Doderecht: Kluwer Academic Publishers, 1993).
6. Kholmatov Sh.Yu., Lakaev S.N., Almuratov F. Bound states of discrete Schrödinger operators on one and two dimensional lattices, J. Math. Anal. Appl. 503:1 (2021), 125280. https://doi.org/10.1016/j.jmaa.2021.125280.
7. Lakaev S.N., Bozorov I.N. The number of bound states of a one-particle Hamiltonian on a three-dimensional lattice. Theoret. and Math. Phys. 158 (2009), 360–376. https://doi.org/10.1007/s11232-009-0030-6.
8. Lakaev S.N., Kholmatov Sh.Yu., Khamidov Sh.I. Bose - Hubbard model with on-site and nearest-neighbor interactions; exactly solvable case. J. Phys. A: Math. Theor. 54 (2021), 245201 [22 pages]. https://iopscience.iop.org/article/10.1088 /1751-8121/abfcf4.
9. Lakaev S.N., Özdemir E. The existence and location of eigenvalues of the one particle Hamiltonians on lattices. Hacettepe J. Math. Stat. 45 (2016), 1693–1703. https://doi.org/10.15672/HJMS.20164515685.
10. Lakaev S.N. Some spectral properties of the generalized Friedrichs model, Journal of Soviet Mathematics, 45(6) (1989), 1540–1563.
11. Lippmann B.A. and Schwinger J. https://doi.org/10.1103/PhysRev.79.469Variational principles for scattering processes. I, Phys. Rev. 79, 469 (1950), 361–379.
12. Mattis D. The few-body problem on a lattice. Rev. Mod. Phys. 58 (1986), 361–379. https://doi.org/10.1103/RevModPhys.58.361.