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INTEGRATING THE LOADED HIROTA EQUATION VIA THE INVERSE SCATTERING METHOD

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Abstract. This study demonstrates that the inverse scattering transform method within the class of rapidly decreasing functions proves highly effective for solving the Cauchy problem for the loaded Hirota equation. The primary focus is on determining the time evolution of scattering data for the Dirac operator, which serves as the solution to the Cauchy problem for the loaded Hirota equation with potentials from the class of rapidly decaying functions. Furthermore, a detailed step-by-step algorithmic procedure for solving this problem has been developed and illustrated with a specific example.

Keywords: Dirac operator, Gelfand-Levitan-Marchenko integral equation, Jost solutions, integrable nonlinear evolution equations, soliton solutions.

Yuklangan Hirota tenglamani sochilish nazariyasining teskari masalalari usulida integrallash

Annotatsiya. Bu ishda sochilish nazariyasining teskari masalalari usuli tez kamayuvchi funksiyalar sinfi doirasida yuklangan Hirota tenglamasiga qo‘yilgan Koshi masalasini yechishda juda samarali ekanligi aniqlangan. Ushbu ishda asosiy e’tibor potentsiali tez kamayuvchi funksiyalar sinfida yuklangan Hirota tenglamasi uchun qo‘yilgan Koshi masalasining yechimi bo‘ladigan Dirak operatorining sochilish nazariyasi berilganlarining vaqt bo‘yicha o‘zgarishni aniqlashga qaratilgan. Bundan tashqari, ushbu masalani hal qilish uchun batafsil bosqichma-bosqich algoritmik jarayon ishlab chiqilgan bo‘lib, buni amalga oshirilishi aniq misolda ko‘rsatilgan.

Kalit so‘zlar: Dirak operatori, Gelfand-Levitan-Marchenko integral tenglamasi, Yost yechimlari, integrallanuvchi nochiziqli evolyutsion tenglamalar, soliton yechimlari.

Интегрирование нагруженного уравнения Хироты с помощью метода обратной задачи рассеяния

Аннотация. В данной работе установлено, что метод обратных задач теории рассеяния в классе быстро убывающих функций является высокоэффективным для решения задачи Коши для нагруженного уравнения Хироты. Основное внимание уделено определению временной эволюции данных рассеяния для оператора Дирака, которые являются решением задачи Коши для нагруженного уравнения Хироты с потенциалом из класса быстро убывающих функций. Кроме того, разработан

подробный пошаговый алгоритмический процесс решения данной задачи, проиллюстрированный на конкретном примере.

Ключевые слова: оператор Дирака, интегральное уравнение Гельфанда-Левитана-Марченко, решения Йоста, интегрируемые нелинейные эволюционные уравнения, солитонные решения.

Introduction. A solitary wave is a localized wave phenomenon consisting of a single peak or trough that propagates in isolation while maintaining its shape, amplitude, and velocity. The Korteweg-de Vries (KdV) equation has been widely recognized as one of the fundamental equations in the study of nonlinear differential equations due to its central role in modeling and analyzing solitary wave dynamics in shallow-water systems. In 1967, Gardner, Greene, Kruskal, and Miura [1] successfully solved the Cauchy problem for the KdV equation on the entire real line by applying the inverse scattering transform (IST) method to the associated Sturm-Liouville operator. This groundbreaking work demonstrated the remarkable effectiveness of spectral methods in solving nonlinear evolution equations.

By 1972, Zakharov and Shabat [2] adapted the inverse scattering method to solve the Cauchy problem for the nonlinear Schrödinger equation, marking another revolutionary breakthrough in the field. Their results not only confirmed the universality of the inverse scattering transform but also provided a strong impetus for the advancement of nonlinear wave dynamics.

Hirota [3] studied the following equation:

$$iu_t + 3i\zeta |u|^2 u_x + \rho u_{xx} + i\mu u_{xxx} + \eta |u|^2 u = 0, \tag{1}$$

where $i^2 = -1$, and the real-valued parameters ζ, ρ, μ, and η satisfy the constraint $\zeta\rho = \mu\eta$. Using the celebrated bilinear Hirota method [4], he derived explicit N-soliton solutions and developed a systematic approach to construct multisoliton solutions.

The above equation (1) can be rewritten in the following form:

$$iu_t + \alpha\left(u_{xx} + 2u|u|^2\right) + i\beta\left(6|u|^2 u_x + u_{xxx}\right) = 0, \tag{2}$$

where the parameters are related as follows: $\zeta = 2\beta$, $\rho = \alpha$, $\mu = \beta$, and $\eta = 2\alpha$

In particular, when $\alpha = 1$ and $\beta = 0$, equation (2) reduces to the nonlinear Schrödinger equation, when $\alpha = 0$ and $\beta = 1$, it transforms into the complex modified Korteweg-de Vries (cmKdV) equation.

The growing interest in optical soliton research is closely tied to their crucial role in fiber-optic communication systems, which enable long-distance information transmission. These systems rely on optical fibers and amplifiers, yet they remain susceptible to noise, signal distortion, and interference. Unlike the classical nonlinear Schrödinger equation, the Hirota equation has found widespread application in studying optical soliton dynamics in fiber waveguides [5, 6]. As a versatile tool for modeling nonlinear processes in scientific and engineering fields, the Hirota equation has inspired numerous solution methods and algorithms, many of which have been thoroughly described in works [7]-[13].

In the works of A.M. Nakhushev [16, 17], a comprehensive definition of loaded equations is presented, with detailed classification of their various types (such as loaded differential equations, loaded integral equations, loaded integro-differential equations, loaded functional equations, and others). Loaded differential equations represent a class of equations that, in addition to differential operators, contain functionals (loads) depending on the values of the solution or its derivatives at specific points or sets. An analysis of scientific literature devoted to loaded equations reveals that works [16]-[20] deserve special attention.

Statement of the Problem. In this study, we focus on analyzing the loaded Hirota equation. Specifically, we consider the following equation:

$$iu_t + P_0(|u(x_0,t)|)(u_{xx} + 2u|u|^2) + iP_1(|u(x_1,t)|)(6|u|^2 u_x + u_{xxx}) + P_2(|u(x_2,t)|)u_x = 0, \tag{3}$$

where the functions $P_i(|u(x_i,t)|)$, $i = 0,1,2$, are assumed to be polynomials in $u(x_i,t)$ and some of its partial derivatives with respect to x and t. Additionally, equation (3) is studied under the following initial condition:

$$u(x,0) = u_0(x). \tag{4}$$

The following constraints must be satisfied:

1. There exists a positive constant $\varepsilon > 0$ such that the inequality

$$e^{\varepsilon|x|}\,|u_0(x)| \le \frac{c}{(1+|x|)^{1+\zeta}}, \quad \zeta > 0, \tag{5}$$

holds, where $c > 0$ is a given constant.

2. The operator $\mathrm{L}(0) = i\begin{pmatrix} \frac{d}{dx} & -u_0(x) \\ -u_0^*(x) & -\frac{d}{dx} \end{pmatrix}$ has exactly N distinct eigenvalues, denoted by $\lambda_1(0), \lambda_2(0), \ldots, \lambda_N(0)$, all of which lie strictly in the upper half-plane $\mathrm{Im}\,\lambda > 0$.

It is assumed that the solution $u(x,t)$ belongs to the class of rapidly decreasing functions, specified by the following properties:

$$M = \left\{ u(x,t):\ \sum_{j=0}^{3} \left| \frac{\partial^j u(x,t)}{\partial x^j} \right| e^{\varepsilon|x|} \le \frac{c}{(1+|x|)^{1+\zeta}}, \quad \varepsilon > 0,\ \ c > 0,\ \ \zeta > 0 \right\}. \tag{6}$$

Preliminaries. In this section, we study the following system of Dirac equations defined on the set of real numbers $x \in (-\infty, \infty)$:

$$\begin{cases} y_{1x} + i\lambda y_1 = u(x) y_2, \\ y_{2x} - i\lambda y_2 = -u^*(x) y_1, \end{cases} \tag{7}$$

The complex-valued potential $u(x)$ satisfies the following conditions:

$$e^{\varepsilon|x|}\,|u(x)| \le \frac{c}{(1+|x|)^{1+\zeta}}, \quad \varepsilon > 0, \tag{8}$$

where c and ζ are positive constants.

Using the operator

$$\mathrm{L} = i\begin{pmatrix} \frac{d}{dx} & -u(x) \\ -u^*(x) & -\frac{d}{dx} \end{pmatrix}, \tag{9}$$

the system of equations (7) can be written as $\mathrm{L}y = \lambda y$. Here $y = (y_1, y_2)^T$.

The scattering theory associated with the operator L has been investigated in detail in [14, 15]. Under the condition (8), the system of equations (7) admits Jost solutions that exhibit the following asymptotic behavior:

$$\left.\begin{aligned}\varphi(x,\lambda)&=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}:\ \begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda x},\quad (\operatorname{Im}\lambda>-\varepsilon/2)\\ \bar{\varphi}(x,\lambda)&=\begin{pmatrix}\bar{\varphi}_1(x,\lambda)\\ \bar{\varphi}_2(x,\lambda)\end{pmatrix}:\ \begin{pmatrix}0\\-1\end{pmatrix}e^{i\lambda x},\quad (\operatorname{Im}\lambda<\varepsilon/2)\end{aligned}\right\}\ \text{as}\ \ x\to-\infty,$$

$$\left.\begin{aligned}\psi(x,\lambda)&=\begin{pmatrix}\psi_1(x,\lambda)\\ \psi_2(x,\lambda)\end{pmatrix}:\ \begin{pmatrix}0\\1\end{pmatrix}e^{i\lambda x},\quad (\operatorname{Im}\lambda>-\varepsilon/2)\\ \bar{\psi}(x,\lambda)&=\begin{pmatrix}\bar{\psi}_1(x,\lambda)\\ \bar{\psi}_2(x,\lambda)\end{pmatrix}:\ \begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda x},\quad (\operatorname{Im}\lambda<\varepsilon/2)\end{aligned}\right\}\ \text{as}\ \ x\to+\infty. \tag{10}$$

Here, the "bar" does not denote complex conjugation (for which the "*" symbol is used instead).

In the strip $|\operatorname{Im}\lambda|<\varepsilon$ there exist these Jost solutions, which form a fundamental system of solutions for the system of equations (10). These solutions satisfy the following relations:

$$\varphi=a(\lambda)\bar{\psi}+b(\lambda)\psi,\quad \bar{\varphi}=-\bar{a}(\lambda)\psi+\bar{b}(\lambda)\bar{\psi}, \tag{11}$$

where

$$\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^T,\ \psi=(\psi_1(x,\lambda),\psi_2(x,\lambda))^T,$$

$$a(\lambda)=W\{\varphi(x,\lambda),\psi(x,\lambda)\}\equiv\varphi_1(x,\lambda)\psi_2(x,\lambda)-\varphi_2(x,\lambda)\psi_1(x,\lambda),$$

$$a(\lambda)\bar{a}(\lambda)+b(\lambda)\bar{b}(\lambda)=1,\quad \bar{a}(\lambda)=a^*(\lambda^*),\quad \bar{b}(\lambda)=b^*(\lambda^*),\ N=\bar{N},\quad \bar{\lambda}_n=\lambda_n^*,$$

$$\bar{\varphi}(x,\lambda)=\begin{pmatrix}\varphi_2^*(x,\lambda^*)\\ -\varphi_1^*(x,\lambda^*)\end{pmatrix},\quad \bar{\psi}(x,\lambda)=\begin{pmatrix}\psi_2^*(x,\lambda^*)\\ -\psi_1^*(x,\lambda^*)\end{pmatrix}.$$

The function $a(\lambda)$, initially defined in the strip $|\operatorname{Im}\lambda|<\varepsilon/2$, admits an analytic continuation to the half-plane $\operatorname{Im}\lambda>-\varepsilon/2$ and has the asymptotics

$$a(\lambda)=1+O\left(|\lambda|^{-1}\right),\quad |\lambda|\to\infty,$$

uniformly in each strip $\operatorname{Im}\lambda\ge\eta$ for $\eta>-\varepsilon/2$. In this case, the function $a(\lambda)$ can have only a finite number of zeros $\lambda_k,\ k=\overline{1,N}$ in the upper half-plane, and these zeros correspond to the eigenvalues of the operator L in the upper half-plane. The eigenvalues of the operator L in the lower half-plane coincide with the zeros of the function $a^*(\lambda^*)$. At such points, the following equalities hold:

$$\varphi(x,\lambda_k)=C_k\psi(x,\lambda_k),\quad \left(\varphi(x,\lambda_k)=\varphi_k,\ \psi(x,\lambda_k)=\psi_k\right),\quad k=\overline{1,N}. \tag{12}$$

The absence of spectral singularities of the operator L means that the functions $a(\lambda)$ and $\bar{a}(\lambda)$ do not have real zeros.

The functions $\psi(x,\lambda)$ and $\bar{\psi}(x,\lambda)$ have the following integral representations:

$$\psi=\begin{pmatrix}0\\1\end{pmatrix}e^{i\lambda x}+\int_x^\infty\Omega(x,s)e^{i\lambda s}ds,\quad \bar{\psi}=\begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda x}+\int_x^\infty\bar{\Omega}(x,s)e^{-i\lambda s}ds, \tag{13}$$

where $\Omega(x,y)=(\omega_1(x,y),\omega_2(x,y))^T$. In the expression (13), the kernel $\Omega(x,y)$ does not depend on the parameter λ and satisfies the relation $u(x)=-2\omega_1(x,x)$.

The kernels $\Omega(x,y)$ and $\bar{\Omega}(x,y)$ satisfy the system of Gelfand-Levitan-Marchenko integral equations:

$$\bar{\Omega}(x,y)+\begin{pmatrix}0\\1\end{pmatrix}\Lambda(x+y)+\int_x^\infty \bar{\Omega}(x,s)\Lambda(s+y)ds=0,$$
$$\Omega(x,y)-\begin{pmatrix}1\\0\end{pmatrix}\Lambda^*(x+y)-\int_x^\infty \Omega(x,s)\Lambda^*(s+y)ds=0, \quad (14)$$

where

$$\Lambda(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\lambda)}{a(\lambda)}e^{i\lambda x}d\lambda-i\sum_{j=1}^{N}C_j e^{i\lambda_j x}, \qquad \bar{\Lambda}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{b}(\lambda)}{a(\lambda)}e^{-i\lambda x}d\lambda+i\sum_{j=1}^{N}C_j e^{-i\lambda_j x}. \quad (15)$$

Definition 1. The following set $\left\{r^+(\lambda),\ \lambda_k,\ C_k,\ k=\overline{1,N}\right\}$ is called the scattering data for the system of equations (7).

Definition 2. A function $r^+(\lambda)=b(\lambda)/a(\lambda)$ is called a function of type (S) in the strip $|\mathrm{Im}\lambda|<\varepsilon/2$ if the following conditions are satisfied:

1) $r^+(\lambda)$ is a meromorphic function in the strip $|\mathrm{Im}\lambda|<\varepsilon/2$, and for any $\eta<\varepsilon/2$, it has the asymptotic behavior

$$r^+(\lambda)=O\left(\lambda^{-1}\right) \quad \text{as} \quad |\lambda|\to\infty,$$

uniformly in the strip $|\mathrm{Im}\lambda|\le\eta$.

2) $r^+(\lambda)$ has no non-real poles in the strip $|\mathrm{Im}\lambda|<\varepsilon/2$ and satisfies the condition

$$1+r^+(\lambda)r^*(\lambda^*)\neq 0,$$

where $r^*(\lambda^*)=\dfrac{b^*(\lambda^*)}{a^*(\lambda^*)}$.

Theorem 1 ([14]). For a function $r^+(\lambda)$ defined in the strip $|\mathrm{Im}\lambda|<\varepsilon/2$, a set of numbers $\lambda_1,\lambda_2,\ldots,\lambda_N$ located in the upper half-plane, and a set of constants $C_1,C_2,\ldots,C_N$ to constitute the scattering data for the operator L of the form (9) with a complex-valued potential satisfying condition (8), the following conditions must hold:

1. The function $r^+(\lambda)$ belongs to class (S) in the strip $|\mathrm{Im}\lambda|<\varepsilon/2$;

2. The functions $\Lambda(x)$ and $\bar{\Lambda}(x)$, constructed from these quantities according to formula (15), satisfy the conditions

$$\left|\Lambda(x)\right|e^{\varepsilon x}\le\frac{c_1}{(1+x)^{1+\zeta}}, \quad \text{for } x\ge a\ (a>-\infty),$$

$$\left|\bar{\Lambda}(x)\right|e^{-\varepsilon x}\le\frac{c_2}{(1+x)^{1+\zeta}}, \quad \text{for } x\le a\ (a<+\infty),$$

for all $a\in\mathrm{R}$.

Theorem 2 ([15]). The scattering data of the operator L uniquely determine L.

It should be noted that the vector functions

$$h_n(x) = \frac{\dfrac{d}{d\lambda}\left(\varphi - C_n\psi\right)|_{\lambda=\lambda_n}}{\dot{a}(\lambda_n)}, \quad n = \overline{1, N}, \tag{16}$$

are solutions of the equation $L(0)h_n = \lambda_n h_n$. Moreover, the functions $h_n(x)$ have the following asymptotics:

$$h_n: \ -C_n\begin{pmatrix}0\\1\end{pmatrix}e^{i\lambda_n x}, \quad \text{as} \ \ x \to -\infty; \quad h_n: \ \begin{pmatrix}1\\0\end{pmatrix}e^{-i\lambda_n x}, \quad \text{as} \ \ x \to \infty. \tag{17}$$

According to the above formulas (17), the following relations hold:

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1}h_{n2} - \varphi_{n2}h_{n1} = -C_n, \quad n = 1, 2, \ldots, N. \tag{18}$$

Evolution of scattering data. Consider the eigenvalue problem

$$\mathrm{L}(t)\varphi = \lambda\varphi, \tag{19}$$

where the potential $u(x,t)$ satisfies equation (3). It should be noted that equation (3) can be expressed in the form

$$\mathrm{L}_t(t) + [\mathrm{L}(t), \mathrm{A} + \mathrm{B}] = i\mathrm{G}, \tag{20}$$

using the following differential operators:

$$\mathrm{A} = P_0(|u(x_0,t)|)\begin{pmatrix} i(|u|^2 - 2\lambda^2) & iu_x + 2\lambda u \\ iu_x^* - 2\lambda u^* & i(-|u|^2 + 2\lambda^2)\end{pmatrix}, \tag{21}$$

$$\mathrm{B} = P_1(|u(x_1,t)|)\begin{pmatrix} -4i\lambda^3 + 2i\lambda|u|^2 - u^*u_x + u_x^*u & 4\lambda^2 u + 2i\lambda u_x - u_{xx} - 2u|u|^2 \\ -4\lambda^2 u^* + 2i\lambda u_x^* + u_{xx}^* + 2u^*|u|^2 & 4i\lambda^3 - 2i\lambda|u|^2 + u^*u_x - u_x^*u\end{pmatrix}. \tag{22}$$

Here,

$$\mathrm{G} = \begin{pmatrix} 0 & G(x,t) \\ G^*(x,t) & 0\end{pmatrix}, \quad G(x,t) = P_2(|u(x_2,t)|)u_x, \ G^*(x,t) = P_2(|u(x_2,t)|)u_x^*$$

and the expression $[\mathrm{L}(t), \mathrm{A}] = \mathrm{L}(t)\mathrm{A} - \mathrm{A}\mathrm{L}(t)$ denotes the commutator of the operators $\mathrm{L}(t)$ and A.

Now, differentiating equation (19) with respect to time and applying the Lax representation (20), we obtain the following relationship:

$$(\mathrm{L}(t) - \lambda\mathrm{I})\left(\varphi_t(x,\lambda,t) - (\mathrm{A} + \mathrm{B})\varphi(x,\lambda,t)\right) = -i\mathrm{G}\varphi(x,\lambda,t). \tag{23}$$

Applying the method of variation of constants, we obtain the equation:

$$\varphi_t(x,\lambda,t) - (\mathrm{A} + \mathrm{B})\varphi(x,\lambda,t) = C(x)\psi(x,\lambda,t) + D(x)\varphi(x,\lambda,t), \tag{24}$$

To determine the unknown functions $C(x)$ and $D(x)$, we derive the following system of equations:

$$\begin{cases} C'(x)\psi_1(x,\lambda,t) + D'(x)\varphi_1(x,\lambda,t) = -G(x,t)\varphi_2(x,\lambda,t), \\ C'(x)\psi_2(x,\lambda,t) + D'(x)\varphi_2(x,\lambda,t) = G^*(x,t)\varphi_1(x,\lambda,t). \end{cases} \tag{25}$$

Here, the prime symbol ($'$) denotes differentiation with respect to x. By solving the system (25) under appropriate boundary conditions, we obtain the following expressions for the coefficients:

$$C'(x) = \frac{1}{a(\lambda,t)}\tilde{\varphi}^T(x,\lambda,t)\mathrm{G}\varphi(x,\lambda,t), \quad D'(x) = -\frac{1}{a(\lambda,t)}\psi^T(x,\lambda,t)\mathrm{G}^T\tilde{\varphi}(x,\lambda,t), \tag{26}$$

where

$$\tilde{\varphi}(x,\lambda,t)=\begin{pmatrix}\varphi_2(x,\lambda,t)\\ \varphi_1(x,\lambda,t)\end{pmatrix}.$$

Now, analyzing equations (21), (22), and (23) as $x\to-\infty$, we arrive at the following results:

$$C(x)\to 0,\quad D(x)\to 2i\lambda^2\left(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|)\right).$$

Consequently, from (26) we derive:

$$C(x)=\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\tilde{\varphi}^T(z,\lambda,t)\mathrm{G}\varphi(z,\lambda,t)dz,$$

$$D(x)=-\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\psi^T(z,\lambda,t)\mathrm{G}^T\tilde{\varphi}(z,\lambda,t)dz+2i\lambda^2\left(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|)\right).$$

As a result, equation (24) can be written in the following form:

$$\varphi_t(x,\lambda,t)-(\mathrm{A}+\mathrm{B})\varphi(x,\lambda,t)=\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\tilde{\varphi}^T(z,\lambda,t)\mathrm{G}\varphi(z,\lambda,t)dz\cdot\psi(x,\lambda,t)+$$

$$+\left[-\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\psi^T(z,\lambda,t)\mathrm{G}^T\tilde{\varphi}(z,\lambda,t)dz+2i\lambda^2(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|))\right]\varphi(x,\lambda,t). \tag{27}$$

Moreover, by substituting the expression from (11), equation (27) takes the following form:

$$a_t(\lambda,t)\bar{\psi}(x,\lambda,t)+b_t(\lambda,t)\psi(x,\lambda,t)-(\mathrm{A}+\mathrm{B})(a(\lambda,t)\bar{\psi}(x,\lambda,t)+b(\lambda,t)\psi(x,\lambda,t))=$$

$$=\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\tilde{\varphi}^T(z,\lambda,t)\mathrm{G}\varphi(z,\lambda,t)dz\cdot\psi(x,\lambda,t)+$$

$$-\frac{1}{a(\lambda,t)}\int_{-\infty}^{x}\psi^T(z,\lambda,t)\mathrm{G}^T\tilde{\varphi}(z,\lambda,t)dz\cdot\left(a(\lambda,t)\bar{\psi}(x,\lambda,t)+b(\lambda,t)\psi(x,\lambda,t)\right)+$$

$$+2i\lambda^2\left(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|)\right)\left(a(\lambda,t)\bar{\psi}(x,\lambda,t)+b(\lambda,t)\psi(x,\lambda,t)\right).$$

By taking the limit as $x\to+\infty$ in the last equality and applying the relations (21) and (22), we arrive at the following results:

$$a_t(\lambda,t)=-\int_{-\infty}^{\infty}\psi^T(z,\lambda,t)\mathrm{G}^T\tilde{\varphi}(z,\lambda,t)dz,$$

$$b_t(\lambda,t)=\frac{1}{a(\lambda,t)}\int_{-\infty}^{\infty}\tilde{\varphi}^T(z,\lambda,t)\mathrm{G}\varphi(z,\lambda,t)dz+ \tag{28}$$

$$-\frac{b(\lambda,t)}{a(\lambda,t)}\int_{-\infty}^{\infty}\psi^T(z,\lambda,t)\mathrm{G}^T\tilde{\varphi}(z,\lambda,t)dz+4i\lambda^2b(\lambda,t)\left(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|)\right).$$

Moreover, since $r^+(\lambda,t)=\dfrac{b(\lambda,t)}{a(\lambda,t)}$, the following formula holds:

$$\frac{dr^+(\lambda,t)}{dt}=4i\lambda^2\left(P_0(|u(x_0,t)|)+2\lambda P_1(|u(x_1,t)|)\right)r^+(\lambda,t)+\frac{1}{a^2(\lambda,t)}\int_{-\infty}^{\infty}\tilde{\varphi}^T(z,\lambda,t)\mathrm{G}\varphi(z,\lambda,t)dz. \tag{29}$$

Lemma 1. Let the vector-functions

$$\varphi(x,\lambda,t)=\begin{pmatrix}\varphi_1(x,\lambda,t)\\ \varphi_2(x,\lambda,t)\end{pmatrix},\quad \psi(x,\lambda,t)=\begin{pmatrix}\psi_1(x,\lambda,t)\\ \psi_2(x,\lambda,t)\end{pmatrix}$$

be a fundamental system of solutions of the system of equations (19). Then the components of these functions satisfy the following equalities:

$$\int_{-\infty}^{\infty} \psi^T(x,\lambda,t)\mathrm{G}^T\tilde{\varphi}(x,\lambda,t)dx = 0, \tag{30}$$

$$\int_{-\infty}^{\infty} \tilde{\varphi}^T(x,\lambda,t)\mathrm{G}\varphi(x,\lambda,t)dx = -2i\lambda P_2(|u(x_2,t)|)a(\lambda,t)b(\lambda,t). \tag{31}$$

Proof. First, we use formulas (6), (10), (11) and (19) to calculate the integral in equation (30):

$$\int_{-\infty}^{\infty} \psi^T(x,\lambda,t)\mathrm{G}\varphi(x,\lambda,t)dx =$$

$$= P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\left[u_x^*(x,t)\varphi_1(x,\lambda,t)\psi_1(x,\lambda,t)+u_x(x,t)\varphi_2(x,\lambda,t)\psi_2(x,\lambda,t)\right]dx =$$

$$= P_2(|u(x_2,t)|)\lim_{d\to\infty}\left[u^*(x,t)\varphi_1(x,\lambda,t)\psi_1(x,\lambda,t)+u(x,t)\varphi_2(x,\lambda,t)\psi_2(x,\lambda,t)\right]\Big|_{-d}^{d} -$$

$$-P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\Big\{u^*(x,t)\left[\varphi_1'(x,\lambda,t)\psi_1(x,\lambda,t)+\varphi_1(x,\lambda,t)\psi_1'(x,\lambda,t)\right]+$$

$$+u(x,t)\left[\varphi_2'(x,\lambda,t)\psi_2(x,\lambda,t)+\varphi_2(x,\lambda,t)\psi_2'(x,\lambda,t)\right]\Big\}dx =$$

$$= \lim_{d\to\infty}P_2(|u(x_2,t)|)\left[-i\lambda(\varphi_1(x,\lambda,t)\psi_2(x,\lambda,t)+\varphi_1(x,\lambda,t)\psi_1(x,\lambda,t))\right]\Big|_{-d}^{d} = 0$$

The integral in (31) is evaluated using a similar approach:

$$\int_{-\infty}^{\infty} \tilde{\varphi}^T(x,\lambda,t)\mathrm{G}\varphi(x,\lambda,t)dx = P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\left[u_x^*(x,t)\varphi_1^2(x,\lambda,t)+u_x(x,t)\varphi_2^2(x,\lambda,t)\right]dx =$$

$$= P_2(|u(x_2,t)|)\lim_{d\to\infty}\left[u^*(x,t)\varphi_1^2(x,\lambda,t)+u(x,t)\varphi_2^2(x,\lambda,t)\right]_{-d}^{d} -$$

$$-2P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\left[u^*(x,t)\varphi_1(x,\lambda,t)\varphi_1'(x,\lambda,t)+u(x,t)\varphi_2(x,\lambda,t)\varphi_2'(x,\lambda,t)\right]dx$$

$$= -2i\lambda P_2(|u(x_2,t)|)a(\lambda,t)b(\lambda,t).$$

Corollary 1. In accordance with Lemma 1 and relations (28) and (30), we obtain $a_t(\lambda,t)=0$, which implies:

$$\lambda_n(t) = \lambda_n(0), \quad n=\overline{1,N}. \tag{32}$$

If we take into account the equalities (29) and (31), we obtain the following formula:

$$\frac{dr^+(\lambda,t)}{dt} = \left(4i\lambda^2 P_0(|u(x_0,t)|)+8i\lambda^3 P_1(|u(x_1,t)|)-2i\lambda P_2(|u(x_2,t)|)\right)r^+(\lambda,t). \tag{33}$$

Differentiating the relation $\varphi_n = C_n\psi_n$ with respect to time and considering equation (32), we obtain the following equality:

$$\frac{dC_n}{dt}\psi_n + C_n\frac{d\psi_n}{dt} = \frac{d\varphi_n}{dt}. \tag{34}$$

Similarly to the continuous spectrum case, for the discrete spectrum we derive the following relation:

$$\frac{\partial \varphi_n}{\partial t} - (\mathrm{A} + \mathrm{B})\varphi_n = \frac{1}{C_n}\int_{-\infty}^{x} \left(G_1\varphi_{n2}^2 + G_2\varphi_{n1}^2\right)dx \cdot h_n$$

$$+\left(\frac{-1}{C_n}\int_{-\infty}^{x}\left(G_2\varphi_{n1}h_{n1} + G_1\varphi_{n2}h_{n2}\right)dx + 2i\lambda_n^2 P_0(|u(x_0,t)|) + 4i\lambda_n^3 P_1(|u(x_1,t)|)\right)\varphi_n.$$

We can write this equation as follows:

$$\frac{dC_n}{dt}\psi_n + C_n\frac{\partial \psi_n}{\partial t} - (\mathrm{A} + \mathrm{B})C_n\psi_n = \frac{1}{C_n}\int_{-\infty}^{x}\left(G_1\varphi_{n2}^2 + G_2\varphi_{n1}^2\right)dx \cdot h_n +$$

$$+\left(\frac{-1}{C_n}\int_{-\infty}^{x}\left(G_2\varphi_{n1}h_{n1} + G_1\varphi_{n2}h_{n2}\right)dx + 2i\lambda_n^2 P_0(|u(x_0,t)|) + 4i\lambda_n^3 P_1(|u(x_1,t)|)\right)C_n\psi_n.$$

(35)

Now, taking into account the expressions (6), (7) and (10), we can have the following:

$$\int_{-\infty}^{\infty}\left(G_2\varphi_{n1}h_{n1} + G_1\varphi_{n2}h_{n2}\right)dx = P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\left(u_x^*\varphi_{n1}h_{n1} + u_x\varphi_{n2}h_{n2}\right)dx =$$

$$= P_2(|u(x_2,t)|)\left(\int_{-\infty}^{\infty}\varphi_{n1}h_{n1}du^* + \int_{-\infty}^{\infty}\varphi_{n2}h_{n2}du\right) = P_2(|u(x_2,t)|)\lim_{d\to\infty}\left(u^*\varphi_{n1}h_{n1} + u\varphi_{n2}h_{n2}\right)\Big|_{-d}^{d} -$$

$$-P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}\left[u^*(\varphi_{n1}'h_{n1} + \varphi_{n1}h_{n1}') + u(\varphi_{n2}'h_{n2} + \varphi_{n2}h_{n2}')\right]dx =$$

$$= -i\lambda_n P_2(|u(x_2,t)|)\int_{-\infty}^{\infty}(\varphi_{n1}h_{n2} + \varphi_{n2}h_{n1})'dx = -2i\lambda_n P_2(|u(x_2,t)|)C_n.$$

(36)

In a similar way, we can derive the following identity:

$$\int_{-\infty}^{\infty}\left(G_1\varphi_{n2}^2 + G_2\varphi_{n1}^2\right)dx = -2i\lambda_n P_2(|u(x_2,t)|)\varphi_{n1}\varphi_{n2}\,|_{-\infty}^{\infty} = 0.$$

(37)

Taking the limit in the above (35) equality ($x \to +\infty$), we obtain the following expression using the (35) equality and the (10), (21), (22), (36), (37) relations:

$$\frac{dC_n}{dt} = \left(4i\lambda^2 P_0(|u(x_0,t)|) + 8i\lambda^3 P_1(|u(x_1,t)|) + 2i\lambda_n P_2(|u(x_2,t)|)\right)C_n.$$

(38)

Hence, the following theorem has been established.

Theorem 3. If the function $u(x,t)$ is a solution to problem (3)-(6), then the time evolution of the scattering data for the non-self-adjoint operator $\mathrm{L}(t)$ with potential $u(x,t)$ is governed by the differential equations (32), (33), and (38).

Remark 1. The time evolution of the scattering data for the operator $\mathrm{L}(t)$ is completely determined by the equations presented in Theorem 3. This provides a rigorous theoretical foundation for solving problem (3)-(6) using the inverse scattering transform method. The resulting set

$$\left\{r^+(\lambda,t),\ \lambda_j(t), C_j(t),\ j = \overline{1,N}\right\}$$

satisfies the conditions of Theorem 1. Therefore, according to Theorems 1 and 2, the potential $u(x,t)$ of the operator $\mathrm{L}(t)$ is uniquely determined and constitutes a solution to problem (3)-(6).

Example 1. Let us consider the initial value problem for a Hirota-type equation with variable coefficients:

$$iu_t + (2t+1)\left(u_{xx} + 2u|u|^2\right) + i\beta(t)u(1,t)\left(u_{xxx} + 6|u|^2 u_x\right) + itu_x = 0,$$

(39)

$$u(x,0) = -\frac{2ib(C_0)^* e^{-2iax}}{|C_0|\cosh 2(bx-\varphi)}, \quad (40)$$

where

$$\beta(t) = \frac{t\exp(12a^2bt^2 - 4b^3t^2 + 8abt + 8abt^2 - bt^2 + 2b)}{2|C_0|} + \frac{t|C_0|\exp(-12a^2bt^2 + 4b^3t^2 - 8abt - 8abt^2 + bt^2 - 2b)}{8b^2},$$

$$\lambda_0 = a+ib,\ b>0, \quad C_0 = |C_0|e^{i\psi}, \quad \frac{|C_0|^2}{4b^2} = e^{4\varphi}. \quad (41)$$

It is straightforward to find that the scattering data for the operator $L(0)$ are:

$$r^+(0) \equiv 0, \quad \lambda_0(0) = a+ib, \quad C_0(0) = C_0.$$

According to Theorem 3, the time evolution of the scattering data is given by:

$$r^+(t) \equiv 0, \quad \lambda_0(t) = \lambda_0(0) = a+ib, \quad C_0(t) = C_0 e^{\mu(t)},$$

where

$$\mu(t) = \int_0^t \left(8i\lambda_0^3\beta(\tau)\,|u(1,\tau)| + 4i\lambda_0^2(2\tau+1) - 2\lambda_0 i\tau\right)d\tau. \quad (42)$$

The kernel $\Lambda(x,t)$ of the Gelfand-Levitan-Marchenko integral equation is determined as:

$$\Lambda(x,t) = -iC_0(t)e^{i\lambda_0 x}. \quad (43)$$

Solving the Gelfand-Levitan-Marchenko integral equation (14) using formula (43), we obtain:

$$\omega_1(x,t) = \frac{iC_0^*(t)e^{-2i\lambda_0^* x}}{1+\left(|C_0|^2/4b^2\right)e^{-4bx}},$$

from which follows:

$$u(x,t) = \frac{-2iC_0^*(t)e^{-2i\lambda_0^* x}}{1+\left(|C_0|^2/4b^2\right)e^{-4bx}}. \quad (44)$$

Setting $x=1$ in the last equality and using formula (42), we arrive at the problem:

$$\mu'(t) = \frac{8i\lambda_0^3\,|C_0^* e^{\mu(t)}|\,e^{2b}\beta(t)}{1+\left(|C_0(t)|^2/4b^2\right)} + 4i\lambda_0^2(2t+1) - 2\lambda_0 it, \quad \mu(0) = 0.$$

The solution to this problem is $\mu(t) = 4i\lambda_0^3 t^2 + 4i\lambda^2(t^2+t) - i\lambda t^2$. Substituting this expression for $\mu(t)$ into formula (44) and using $C_0 = |C_0|\,e^{i\psi}$, we find the solution to the Cauchy problem (39)-(40):

$$u(x,t) = -\frac{2ib\exp\left\{(-4ia^3+12iab^2)t^2 + (-4ia^2+4ib^2)(t^2+t) + iat^2 - 2iax - i\psi\right\}}{\cosh\left(2bx - 2\varphi - 4b(-3a^2+b^2)t^2 + 8ab(t^2+t) - bt^2\right)}.$$

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