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ON NUMBERS REPRESENTABLE AS THE SUM OF FOUR SQUARES OF PRIME NUMBERS

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Abstract. In this work, an new estimate is obtained for the number of representations of a sufficiently large natural number as a sum of squares of four prime numbers. Additionally, the cardinality of the set of natural numbers that cannot be represented in this form (the exceptional set of the problem) is estimated. The obtained result, in a certain sense, improves and complements previously known results in this field. The proof of the main result utilizes the Hardy–Littlewood circle method in the variant of I.M. Vinogradov.

Keywords: circle method, estimate, exceptional set, number of representations, prime numbers, trigonometric sum.

To‘rtta tub sonning kvadratlari yig‘indisi ko‘rinishida ifodalanuvchi sonlar haqida

Annotatsiya. Ishda berilgan yetarlicha katta natural sonni to‘rtta tub sonlarning kvadratlari yig‘indisi ko‘rinishida ifodalashlar soni uchun quyidan yangi baho olingan va shunday ko‘rinishda ifodalash mumkin bo‘lmagan natural sonlar to‘plami(masalani maxsus to‘plami)ning quvvati baholangan. Olingan natija ilgari shu sohada olingan natijalarni ma’lum ma’noda yaxshilaydi va to‘ldiradi. Asosiy natijani isbotlashda Xardi-Littlvudning doiraviy usulidan I.M.Binogradov variantida foydalaniladi.

Kalit so‘zlar: baho, doiraviy usul, ifodalashlar soni, maxsus to‘plam, tub sonlar, trigonometrik yig‘indi.

О числах представимых в виде суммы квадратов четырех простых чисел

Аннотация. В работе доказана, новая оценка для количества представлений заданного достаточно большого натурального числа в виде суммы квадратов четырех простых чисел, а также оценена мощность множества натуральных чисел, которые нельзя представить в указанном виде (исключительное множество задачи). Полученный результат в определенном смысле улучшает и дополняет ранее известные результаты в этой области. При доказательстве основного результата используется круговой метод Харди–Литтлвуда в варианте И.М. Виноградова.

Ключевые слова: исключительное множество, количество представлений, круговой метод, оценка, простые числа, тригонометрическая сумма.

1. Formulation of the result. It is known that Hua-Lo-Kena (see p. 110, [1]) proved that every sufficiently large integer can be represented as the sum of five squares of prime numbers. Information on the results obtained in this direction in recent years can be found in the works of Y. H. Wang [2], I. Allakov, N. S. Muzropova [3]. Given Hua-Lo-Ken's result and J. L. Lagrange's statement about the sum of four squares, it is reasonable to assume that every sufficiently large number can be represented as a sum of four squares of prime numbers, that is, in the form

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 \tag{1}$$

where p_1, p_2, p_3, p_4 are prime numbers.

Despite the fact that there are numerous works in the direction of solving this problem (see [4]), it remains unsolved. In 2000, J. Y. Liu and M. C. Liu [5] proved that if $N \geq 2$ and $U(N)$ is the set of positive integers $n \equiv 4(\mathrm{mod}\,24)$, $n \leq N$ that cannot be represented in the form (1), then the estimate

$$E(N) = cardU(N) = N^{\theta}, \text{ where } \theta > 13/15. \tag{2}$$

holds.

Previously, Hua [1] and W. Schwarz [6] proved that, $E(N) = N\log^{-A} N$ for any $A > 0$. Note also that P. Shields [7], V.A. Plaksin [8] and E.B. Kovalchik [9] obtained an asymptotic formula in this problem, which is valid for all $n \equiv 4(\mathrm{mod}\,24)$, $n \leq N$ except for no more than $E(N) < N\exp\left(-c_1\sqrt{\log N}\right)$ of their values, where c_1 is an absolute constant.

Let, further $L = \log N$, $C(N) = \dfrac{c_2}{\log\log N}$, $c_1, c_2, \ldots$ be absolute constants, $e(n\alpha) = e^{2\pi i n\alpha}$, α be a real number.

Let's put

$$r(n) = \sum_{\substack{n = p_1^2 + \ldots + p_4^2 \\ p_i^2 \leq N,\, i=1,4}} (\log p_1)\mathrm{L}\,(\log p_4), \tag{3}$$

$$R(n) = \sum_{\substack{n = p_1^2 + \ldots + p_4^2 \\ p_i^2 \leq N,\, i=1,4}} 1. \tag{4}$$

Then, by virtue of (3) and (4), we have

$$r(n) \leq \frac{1}{16}(\log^4 N)R(n). \tag{5}$$

Therefore

$$R(n) \geq \frac{16}{\log^4 N} r(n). \tag{6}$$

From (5) and (6) it follows that if $r(n) > 0$, then for n there is a representation in the form (1).

In the present work, we will prove the following theorem.

Theorem 1. If N is a sufficiently large number then,

$$E(N) = N^{13/15 + 4C(N)}, \tag{7}$$

and for $n \notin U(n)$

$$r(n) = \frac{\pi^2}{16}\mathfrak{S}(n)n + O\left(NL^{-1}\right), \tag{8}$$

where $\mathfrak{S}(n)$? 1 is the singular series of the problem.

Corollary. For $R(n)$ the number of representations of a given $N/2 < n \leq N$, $n \equiv 4(\mathrm{mod}\,24)$ in the form (1) the estimate is valid

$$R(n) > c_2 \frac{n}{\log^4 n}\left(1 - c_3\frac{1}{\log n}\right)$$

The obtained result is an improvement of previously obtained results [5-9].

2. Notation and brief description of the method. Let us prove the formulated theorem the Hardy-Littlewood circular method [10].We will prove the formulated theorem the Hardy-Littlewood circular method [10]. In this case, will use a similar scheme, which was used by us in our work [11].

To do this, we need the following notation: $\varphi(n)$, $\mu(n)$ and $\Lambda(n)$ denote the Euler, Mobius and Mangoldt functions, respectively. $d(n)$ is the number of divisors of a natural number n, and $d_v(n)$ is

the generalized divisor function, i.e. the number of solutions to the equation $n = n_1 n_2 \text{L}\ n_\nu$ in natural numbers. By $(a,b,...)$ and $[a,b,...]$ we denote the GCD and LCM of numbers $a,b,...$.

Let $A \geq 16$

$$P = N^{2/15}/L^A, \quad Q = N/\left(PL^{14}\right). \tag{9}$$

Dirichlet's theorem on rational approximation, each $\alpha \in \left[Q^{-1}, 1+Q^{-1}\right]$ can be represented as

$$\alpha = aq^{-1} + \eta, \quad |\eta| \leq (qQ)^{-1}, \tag{10}$$

where a,q are integers with the condition $1 \leq a \leq q \leq Q$ and $(a,q)=1$. The interval $\alpha \in \left[Q^{-1}, 1+Q^{-1}\right]$ is we divide into two parts as follows. Let $\mathfrak{M}(a,q)$ be a set α satisfying condition (10) and $q \leq Q$. Then the sets $\mathcal{M}$ and $\mathcal{M}'$ are defined as follows:

$$\mathcal{M} = \bigcup_{q \leq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(a,q), \quad \mathcal{M}' = \left[Q^{-1}, 1+Q^{-1}\right] \setminus \mathcal{M}. \tag{11}$$

$\mathcal{M}$ and $\mathcal{M}'$ we will called it a major arc, and $\mathcal{M}'$ the minor arc. Since $2P \leq Q$, the intervals $\mathfrak{M}(a,q)$ and $\mathfrak{M}(a',q')$ do not intersect. Let N be a sufficiently large integer and $N/2 < n \leq N,\ n \equiv 4 (\text{mod}\, 24)$. Let's put

$$S(\alpha) = \sum_{p^2 \leq N} (\log p) e\left(\alpha p^2\right). \tag{12}$$

Then we have

$$r(n) = \int_0^1 S^4(\alpha) e(-n\alpha)\, d\alpha \tag{13}$$

Indeed, by virtue of (3) and (12) we have

$$\int_0^1 S^4(\alpha) e(-n\alpha)\, d\alpha = \int_0^1 \sum_{p_1^2,...,p_4^2 \leq N} (\log p_1) \text{L}\ (\log p_4) e\left(\alpha\left(p_1^2 + ... + p_4^2 - n\right)\right) d\alpha =$$

$$= \sum_{p_1^2,...,p_4^2 \leq N} (\log p_1) \text{L}\ (\log p_4) \int_0^1 e\left(\alpha\left(p_1^2 + ... + p_4^2 - n\right)\right) d\alpha = r(n).$$

By virtue of (11), the integral on the right side of equality (13) can be represented

$$r(n) = \int_0^1 S^4(\alpha) e(-n\alpha)\, d\alpha = \left(\int_{\mathcal{M}} + \int_{\mathcal{M}'}\right) S^4(\alpha) e(-n\alpha)\, d\alpha = r_1(n) + r_2(n), \tag{14}$$

where

$$r_1(n) = \int_{\mathcal{M}} S^4(\alpha) e(-n\alpha)\, d\alpha, \quad r_2(n) = \int_{\mathcal{M}'} S^4(\alpha) e(-n\alpha)\, d\alpha.$$

Here $r_1(n)$ is the main term and $r_2(n)$ is the remainder.

3. Estimation of the integral over a minor arc. To estimate the contribution of the minor arcs, we note that each $\alpha \in \mathcal{M}'$ a can be written in the form (10), for which condition

$$P < q \leq Q, \quad 1 \leq a \leq q, \quad (a,q) = 1. \tag{15}$$

holds.

Applying theorem 2 of A.Ghosha [12] (see also [13]), we find

$$S(\alpha) = N^{1/2+C(N)}\left(q^{-1} + N^{-1/4} + qN^{-1}\right)^{1/4} \text{ where } C(N) = \frac{c_2}{\log\log N}.$$

From here, by virtue of condition (15) and (9), we obtain

$$S(\alpha) = N^{1/2+C(N)}\left(P^{-1} + N^{-1/4} + QN^{-1}\right)^{1/4} = N^{1/2+C(N)} P^{-1/4}\left(1 + \frac{P}{N^{1/4}} + \frac{QP}{N} qN^{-1}\right)^{1/4} = N^{1/2+C(N)} P^{-1/4}, \tag{16}$$

since

$$\frac{P}{N^{1/4}}=\frac{N^{\frac{2}{15}-\frac{1}{4}}}{L^A}=\frac{N^{-\frac{7}{15}}}{L^A}<1,\qquad \frac{QP}{N}=\frac{N}{L^{14}N}=\frac{1}{L^{14}}<1.$$

Also we have

$$\int_0^1\left|S(\alpha)\right|^4 d\alpha=\int_0^1\left|S(\alpha)\overline{S(\alpha)}\right|^2 d\alpha=$$

$$\sum_{p_i^2\le N,\ i=\overline{1,4}}(\log p_1)\mathrm{L}\ (\log p_4)\int_0^1 e\left(\left(p_1^2+p_2^2-p_3^2-p_4^2\right)\alpha\right)d\alpha\le(\log N)^4\sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2\\ p_i^2\le N,\ i=\overline{1,4}}}1.$$

The sum on the right side of the last inequality, by virtue of (6.10) of work [5], is estimated as $O\left(NL^2\right)$, therefore

$$\int_0^1\left|S(\alpha)\right|^4 d\alpha=\ NL^6. \tag{17}$$

Therefore, applying Bessel's inequality and using estimates (16) and (17), we obtain $\sum_{N/2<n\le N}\left|\int_{\mathcal{M}'}S^4(\alpha)e(-n\alpha)d\alpha\right|^2=\int_{\mathcal{M}'}\left|S(\alpha)\right|^8 d\alpha=\left\{\max_{\alpha\in\mathcal{M}'}\left|S(\alpha)\right|^4\right\}\int_0^1\left|S(\alpha)\right|^4 d\alpha=$

$$=\left(N^{\frac{1}{2}+C(N)}P^{-\frac{1}{4}}\right)^4 NL^6=\ N^{3+4C(N)}P^{-1}. \tag{18}$$

4. **Research of the integral $r_1(n)$ and completion of the proof of the theorem.** For an arbitrary character $\chi(\bmod q)$ and the principal character $\chi^0(\bmod q)$ (regarding the properties of Dirichlet characters, see p. 35 of H. Davenport's book [14]), the sums $C(\chi,a)$ and $C(q,a)$ will be defined as follows

$$C(\chi,a)=\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{ah^2}{q}\right),\qquad C(q,a)=C\left(\chi^0,a\right).$$

Here $\overline{\chi}$ is the Dirichlet character conjugate with the character $\chi(\bmod q)$.

If $\chi_1,...,\chi_4$ are characters modulo q, let

$$B(n,q,\chi_1,...,\chi_4)=\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(-\frac{an}{q}\right)C(\chi_1,a)\mathrm{L}\ C(\chi_4,a)$$

$$B(n,q)=B\left(n,q,\chi^0,...,\chi^0\right)$$

and

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\frac{B(n,q)}{\varphi^4(q)} \tag{19}$$

By virtue of Theorem 2 of [5], for $2\le n<N$ we have

$$\int_{\mathcal{M}}S^4(\alpha)e(-n\alpha)d\alpha=\frac{\pi^2}{16}\mathfrak{S}(n)n+O\left(NL^{-1}\right). \tag{20}$$

Here $\mathfrak{S}(n)$ is defined as in equality (19) and according to Lemma 3.2 (ii) of work [5] $n\equiv 4(\bmod 24)$ satisfies the relation $c_4<\mathfrak{S}(n)=\ (\log\log n)^{11}$, and if $n\not\equiv 4(\bmod 24)$, then $\mathfrak{S}(n)=0$.

Therefore from (14) and (20) we get

$$r(n)=\frac{\pi^2}{16}\mathfrak{S}(n)n+O\left(NL^{-1}\right)+r_2(n) \tag{21}$$

By virtue of (18) we have

$$\sum_{N/2<n\le N}\left|r_2(n)\right|^2 = N^{3+4C(N)}P^{-1}.$$

It follows that the number $n \equiv 4 (\mathrm{mod}\, 24)$ and $N/2 < n \le N$ for which $r_2(n) \ge NL^{-1}$ is no more than

$$= \frac{N^{3+4C(N)}P^{-1}}{N^2L^{-2}} = N^{1+4C(N)}P^{-1}L^2 = N^{1+4C(N)}N^{-2/15}L^{A+2} = N^{13/15+4C(N)}L^{A+2}, \qquad (22)$$

where $A \ge 16$ и $C(N) = \frac{c_2}{\log\log N}$. For the remaining n the inequality holds

$$r_2(n) < NL^{-1}. \qquad (23)$$

Thus, from (21) and (22) we find that

$$r(n) = \frac{\pi^2}{16}\mathfrak{S}(n)n + O\left(NL^{-1}\right),$$

for all $n \equiv 4 (\mathrm{mod}\, 24)$ and $N/2 < n \le N$ except for no more than values of them $= N^{13/15+4C(N)}L^{A+2}$.
Thus, Theorem 1 is proved.

Since, if $\varepsilon > 0$ is an arbitrary fixed constant, then for a sufficiently large N we have

$$= N^{13/15+4C(N)}L^{A+2} < N^{13/15+11\varepsilon}.$$

Therefore, the obtained result is better than the results of work [5].
Proof of the corollary. From inequality (6) and equality (14) we find

$$R(n) \ge \frac{16}{\log^4 N}\left(r_1(n)+r_2(n)\right) > \frac{16}{\log^4 N}\left(r_1(n)-\left|r_2(n)\right|\right).$$

Substituting the corresponding estimates for $r_1(n)$ and $r_2(n)$ from (21) and (23) into the right-hand side of the last inequality, we obtain the statement given in the corollary.

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