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Obuna indeksi – yakka tartibdagi obunachilar uchun **- 5583,**
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MUNDARIJA	
Кулжанов У.Н., Адилов А.А. *Спектралные свойства одночастичного оператора шредингера с контактным потенциалом*……..	3-7
Eshbekov R. *The cauchy problem for the modified korteweg-de vries equation with finite density in the class of periodic functions*……..	8-16
Abdurakhmonov B., Zaripov O., Teshayeva M. *Lagranj ko`paytuvchilari yordamida tengsizliklarni isbotlash*……	17-21
Umirzoqova I., Boboxonova G. *Oshkormas funksiya egriligi va kvadratik formalari*……..	22-26
Qurbonov H.G‘., Rustamov M.K., Gafurova D.A., Temiron N.N., Toshpulatova M.Sh. *Synthesis and properties of difficult combustion material based on modified polyacrylonitrile*……	27-32
Холиков Д.Ш., Худайбердиев З.Б., Исмоилов Х.Ф. *Упрощение нелинейного уравнения крутильных колебаний круглого конического слоя*……	33-38
Kamoliddinov D.U. *Density and regular distribution*……	39-43
Kuliev K., Eshimova M., Turaqulov T. *A new method for norm of a hardy-volterra operator*……	44-50
Kuziev B.N. *A comparative study on feature selection methods in QSAR*……..	51-55
Мардонов Ж.А. *Задача коши для неоднородного лапласова поля*……	56-61
Нуритдинов И., У.Эсанов З., Саидахмедов К.Х., Мажидов Х.М. *Преобразовании Dy1 центров в Dy2 центры в кристаллах* Dy_2SiO_5……	62-66
Akhmadova M.O. *The number of eigenvalues of the discrete schrödinger operator*……	67-71
Курбаниязов А.А., Сырлыбекқызы С., Раманкулов Ш. Ж., Аймбетова И.О., Ещанова Т.Д. *Новые данные высокого временного разрешения о морских и их изменчивости в зависимости от климатических условий*……	72-77
Ikromov I.A., Safarov A.R., Khudoyberdiev D. *Invariant estimates for generalized oscillatory integrals with polynomial phase*……	78-86
Urazbayev A.K. *Amudaryo hozirgi deltasidagi yer usti suv oqimlarining geografik strukturalari*……	87-91
Komilova N.K., Karshibayeva L.K. *Iqlim o‘zgarishlari va aholi kasalliklarining hududiy tarkibi (Jizzax va Sirdaryo viloyatlari misolida)*……	92-99
Нуритдинов И., Эсанов З.У., Саидахмедов К.Х., Ирисов Ш. *Влияние ионизирующего излучения на спектрально-люминесцентные свойства органических сцинтилляторов ups-923a*……	100-104
Alimkulov N.R. *Antropogen landshaftlarni tadqiq etish asosida landshaft-ekologik sharoit tushunchasining shakllanishi*……	105-113
Кувандиков О.К., Усаров У.Т. *Электрические, гальваномагнитные свойства редкого металла при различных температурах*	114-125
Komilova N.K., Ravshanov A.X. *Global iqlim o‘zgarishlari sharoitida O‘zbekistonda yuqumli-parazitar kasalliklar tarqalishining hududiy jihatlari*……	126-132

Рахматуллаев И.А., Ботиров Х.З., Чернега Н.В., Хайдаров Х.С. *Двухфотонно-возбуждаемая люминесценция в полициклических ароматических соединениях.....*	133-138
Anarova Sh.A., Jabbarov J.S., Rustamov A.A. Kattalashtirish masshtabiga bog'liq bo'lmagan fraktal o'lchovlarni geometrik modellashtirish…….	138-142
Nurmuhamedova D.K., Kutlieva G.J., Sohibnazarova X.A., Zairova X.T. *Mahalliy sut achituvchi bakteriyalarning antimikrob va bakteriosinogen faolligini tadqiq etish …….*	143-148
Тайланов Н.А., Уринов С.Х. *Численное моделирование уравнения гросса-питаевского.....*	149-159

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INVARIANT ESTIMATES FOR GENERALIZED OSCILLATORY INTEGRALS WITH POLYNOMIAL PHASE

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Abstract. In this paper we consider the problem on uniform estimates for generalized oscillatory integrals given by Mittag- Leffler functions with the polynomial phase. We obtain invariant estimates for corresponding integrals.

Keywords: Mittag-Leffler functions, phase function, amplitude, oscillatory integrals.

Fazasi ko‘phad bo‘lgan umumlashgan tebranuvchan integrallarning invariant baholari

Annotatsiya. Ushbu maqolada biz ko’phad fazali Mittag-Leffler funksiyalari bilan berilgan umumlashgan tebranuvchan integrallarning tekis baholari masalasini ko'rib chiqamiz. Ushbu integrallar uchun invariant baholarni olamiz.

Kalit so‘zlar: Mittag-Leffler funksiyalari, fazaviy funksiya, amplituda, tebranish integrallari.

Инвариантные оценки для обобщенных осциллирующих интегралов с полиномиальной фазой

Аннотация. В данной работе рассматривается задача о равномерных оценках для обобщенных осциллирующих интегралов, заданных функциями Миттаг-Леффлера с полиномиальной фазой. Получен инвариантные оценки для соответствующих интегралов.

Ключевые слова: Функции Миттаг-Леффлера, фазовая функция, амплитуда, осцилляторные интегралы.

Introduction. Many problems of harmonic analysis, analytic number theory and mathematical physics involve trigonometric (oscillatory) integrals with polynomial phase, a common problem of integration of rational polynomials instead of integrating arbitrary functions [1, 2, 3, 4, 7, 8]. One of the main tools in harmonic analysis for estimation of oscillatory integrals is van der Corput lemma which was established in one dimensions in [14], while a multidimensional version have been obtained in [11] using combining the standard one-dimensional van der Corput lemma with the stationary phase method. Some estimates for oscillatory integrals with polynomial phase can be found, for instance, in [4, 5, 10].

The function $E_\alpha(z)$ is named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846-1927) who defined it by a power series

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha\in\mathbb{C}, Re(\alpha)>0, \tag{1}$$

and studied its properties in [15]-[16] in connection with his summation method for divergent series.

A classic generalizations of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta\in\mathbb{C}, Re(\alpha)>0, \tag{2}$$

have been deeply investigated, for instance, in [17]–[18].

In this paper we replace exponential function with the Mittag-Leffler-type function and study the multidimensional generalization of the oscillatory integrals. Namely the main objective of this paper is to obtain estimates for the following oscillatory integral

$$I_{\alpha,\beta}(a)=\int_{Q^n}E_{\alpha,\beta}(iP(a,x))\psi(x)dx, \tag{3}$$

where $0<\alpha<1$, $\beta>0$, $P(a,x)$ is a phase, $\psi\in C^{\infty}(\mathrm{R}^n)$ is an amplitude and $Q^n:=[0,1]^n$ is n dimensional cube. Note that in case $\alpha=1$ and $\beta=1$ we reduse to the case of classical oscillatory integral.

In [12] and [13] analogues of the van der Corput lemmas involving Mittag-Leffler functions for one dimensional integrals have been studied. We consider estimates for multidimensional generalization of the oscillatory integrals with polynomial phase. This work is analogous to [7] and is an application for oscillatory integrals with Mittag-Leffler functions. An analogous problem on estimates for Mittag-Leffler functions with the smooth phase functions of two variables having simple singularities was considered in [6].

Proposition 1. If $0<\alpha<2,\beta$ is an arbitrary real number and μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, then there is a $C>0$ such that

$$|E_{\alpha,\beta}(z)|\le\frac{C}{1+|z|},z\in\mathrm{C},\mu\le|\arg(z)|\le\pi. \tag{4}$$

Note that there is relation between the behavior of the classical oscillatory integrals and measure of the sublevel set of the corresponding phase function. More precisely, the following statement holds true.

Invariant estimates for homogeneous polynomial phase

We consider the generalized oscillatory integrals with phase function which is a homogeneous polynomial of degree three in two variables

$$P_3(a,x,y)=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3, \tag{5}$$

here $a:=(a_0,a_1,a_2,a_3)\in\mathrm{R}^4$ are coefficients. Consider

$$I_{\alpha,\beta}(a)=\int_{\mathrm{R}^2}E_{\alpha,\beta}(iP_3(a,x,y))\psi(x,y)dxdy, \tag{6}$$

where ψ is a smooth function with compact support.

We study behavior of the integral (6) in the case when the coefficients of the polynomial tend to infinity. Let us obtain estimates for integral (6) in terms of the invariants of the group of motions of the Euclidean plane. Note that, the discriminant of a polynomial denoted by D is defined by the formula:

$$D=3a_1^2a_2^2+6a_0a_1a_2a_3-4a_0a_2^3-4a_1^3a_3-a_0^2a_3^2.$$

It is the invariant of the group $SL(2,\mathrm{C})$.

Theorem 1. *For the integral (3) with phase (5) and an amplitude function ψ with compact support the following inequality*

$$|I_{\alpha,\beta}(a)|\le\frac{c\|\psi\|_{C(\mathrm{R}^n)}}{|D|^{\frac{1}{6}}}$$

holds. Here c is a constant independent of $a\neq0$ and $D\neq0$ is discriminant of the polynomial $P_3(a,x,y)$.

Auxiliary estimates

In this section we consider one-dimensional integrals. The results of this section will be used in the next section to get invariant estimates. Now we consider the following example for $d=3$.

Let $0<\delta<1$. We consider the integral

$$J_{\alpha,\beta}(p,q)=\int_{\mathrm{R}}E_{\alpha,\beta}(i(x^3+px+q))dx, \tag{7}$$

where $0<\alpha<1$, $\beta>0$.

Theorem 2. *1) Let* $\frac{1}{3} < \delta < \frac{1}{2}$. *Then*

$$\left|J_{\alpha,\beta}(p,q)\right| \le \frac{c_\delta}{\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{3\delta-1}{6}}}, \tag{8}$$

for any $(p,q) \neq (0,0)$.

2) Let $\frac{1}{2} < \delta < 1$. *Then*

$$\left|J_{\alpha,\beta}(p,q)\right| \le \frac{c_\delta}{|D|^{\delta-\frac{1}{2}}\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{2-3\delta}{6}}}, \tag{9}$$

where $D = \frac{p^3}{27}+\frac{q^2}{4} \neq 0$ is discriminant of the polynomial x^3+px+q and $c_\delta > 0$ is a positive number depending only on δ.

Proof of Theorem 2. If $p = q = 0$, then the required estimate is trivially satisfied. We use the inequality (4) for the integral (11) and obtain

$$|J_{\alpha,\beta}| \le \int_R \frac{dx}{1+|x^3+px+q|} \le \int_R \frac{dx}{|x^3+px+q|^\delta} := J(p,q), \tag{10}$$

where $0 < \delta \le 1$. Indeed, if $|x^3+px+q| \le 1$ then for any $\delta > 0$ we have

$$1+|x^3+px+q| \ge 1 \ge |x^3+px+q|^\delta .$$

Conversely, if $|x^3+px+q| > 1$ then since $0 < \delta \le 1$ we obtain:

$$1+|x^3+px+q| \ge |x^3+px+q| \ge |x^3+px+q|^\delta .$$

Thus, combined the obtained estimates we get:

$$\frac{1}{1+|x^3+px+q|} \le \frac{1}{|x^3+px+q|^\delta}.$$

Finally, integrating the last inequality we arrive at the estimate (10).

Let $\frac{1}{3} < \delta < \frac{1}{2}$ and $(p,q) \neq (0,0)$. We consider the following cases separately.

$$\max\left\{\frac{|p|^3}{27}, \frac{q^2}{4}\right\} = \frac{|p|^3}{27} \tag{11}$$

and

$$\max\left\{\frac{|p|^3}{27}, \frac{q^2}{4}\right\} = \frac{q^2}{4}. \tag{12}$$

Suppose the condition (11) holds. Then using the change of variables $x = |p|^{\frac{1}{2}} y$ we obtain

$$|J(p,q)| = \int_R \frac{|p|^{\frac{1}{2}} dy}{\left||p|^{\frac{3}{2}} y^3 + |p|^{\frac{1}{2}} py + q\right|^\delta} = \frac{1}{|p|^{\frac{3\delta-1}{2}}} \int_R \frac{dy}{\left|y^3 + sgn(p)y + q|p|^{-\frac{3}{2}}\right|^\delta}.$$

Let us consider the integral

$$|J_1(p,q)| = \int_R \frac{dy}{|y^3 + sgn(p)y + B|^\delta}, \tag{13}$$

where $|B| = \left|\frac{q}{|p|^{\frac{3}{2}}}\right| \le \frac{2}{3\sqrt{3}}$. We show that this integral is bounded, when $\frac{1}{3} < \delta < \frac{1}{2}$. Indeed, if $|y| \ge 2$, then

$$|y|^3\left|1+\frac{1}{y^2}+\frac{B}{y^3}\right| \ge |y|^3\left(1-\frac{1}{y^2}-\frac{|B|}{|y|^3}\right) \ge |y|^3\left(1-\frac{1}{4}-\frac{1}{12\sqrt{3}}\right) \ge \frac{|y|^3}{2}.$$

Consequently,

$$|J(p,q)| = \int_{-2}^{2} \frac{dy}{|y^3 + sgn(p)y + B|^\delta} + R(B), \tag{14}$$

where $R(B)$ is a bounded function of $B \in \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.

Let

$$|J_{10}| := \int_{-2}^{2} \frac{dy}{|y^3 + sgn(p)y + B|^\delta}. \tag{15}$$

Suppose $sgn(p) = 1$. Then the polynomial $y^3 + y + B$ has no multiple roots. In fact, for any B it has one simple root. Thus,

$$|J_0| = \int_{-2}^{2} \frac{dy}{|y^3 + y + B|^\delta} \le c,$$

for $\delta < \frac{1}{2} < 1$.

Suppose $sgn(p) = -1$. Then, since B belong to a compact set, it suffices to obtain the corresponding "local" estimates. Let ε be a sufficiently small positive number. Consider two sets

$$A_1 = \left\{\left|B^2 - \frac{4}{27}\right| \ge \varepsilon\right\}, A_2 = \left\{\left|B^2 - \frac{4}{27}\right| \le \varepsilon\right\}.$$

First, consider the estimate for the integral when B lies in the set A_1. In this case equation $y^3 - y + B = 0$ has three different roots $y_1(B), y_2(B), y_3(B)$, since $|y_j(B) - y_k(B)| \ge \Delta(\varepsilon) > 0$ when $j \ne k$. Since $\delta < 1$, then the following integral convergences

$$J_{10} := \int_{-2}^{2} \frac{dy}{|(y - y_1(B))(y - y_2(B))(y - y_3(B))|^\delta}. \tag{16}$$

Since B belongs to the compact set A_1, the integral (16) is uniformly bounded for $\delta < 1$. Therefore, we have

$$|J| = \frac{1}{|p|^{\frac{3\delta-1}{2}}} \int_{-|p|^{\frac{-1}{2}}}^{|p|^{\frac{1}{2}}} \frac{dy}{|y^3 - y + B|^\delta} \le \frac{c_\delta}{|p|^{\frac{3\delta-1}{2}}},$$

as $B \in A_1$. Now we consider the integral in the case when $B \in A_2$. For the sake of definiteness, we can assume that $\left|B - \frac{2}{3\sqrt{3}}\right| < \varepsilon$, where $0 < \varepsilon < \frac{1}{6\sqrt{3}}$ a fixed positive number. Note that

$y_{1,2} = \pm\frac{1}{\sqrt{3}}$ are critical points of the function $F(y) = y^3 - y + B.$ Thus, the number $B = B_0 = \frac{2}{3\sqrt{3}}$ for the function $F(y, B_0)$ is a simple critical value. Let us study behavior of the integral as $|B - B_0| < \Delta,$ where Δ is sufficiently small fixed positive number. Since $F\left(\frac{1}{\sqrt{3}}\right) = 0,$ then $F(y, B_0) = \left(y - \frac{1}{\sqrt{3}}\right)^2\left(y + \frac{2}{\sqrt{3}}\right)$. Hence,

$$F\left(y + \frac{1}{\sqrt{3}}, B\right) = y^3 + \sqrt{3}y^2 - \frac{2}{3\sqrt{3}} + B = y^2\left(y + \sqrt{3}\right) + H,$$

where $H = B - \frac{2}{3\sqrt{3}}$.

We consider the following integral

$$\int_{-\sigma}^{\sigma} \frac{dy}{\left|y^2\left(y + \sqrt{3}\right) + H\right|^{\delta}}, \tag{17}$$

where $\sigma > 0$ sufficiently small fixed positive number. If $\frac{1}{3} < \delta < \frac{1}{2},$ then according to the [7], the integral (17) is uniformly bounded with respect to H. This completes the proof of the first part of Theorem 2.

Next, suppose that $\frac{1}{2} < \delta < 1.$ We use change of variables as $z = y\sqrt{y + \sqrt{3}}$ and and denoting the inverse function by $y = y(z),$ we have

$$\int_{-\sigma}^{\sigma} \frac{dy}{\left|y^2\left(y + \sqrt{3}\right) + H\right|^{\delta}} = \int_{-\sigma_1}^{\sigma_2} \frac{y'(z)dz}{\left|z^2 + H\right|^{\delta}} =$$

$$= \frac{1}{\sqrt{3}}\int_{-\sigma_1}^{\sigma_2} \frac{dz}{\left|z^2 + H\right|^{p}} + c\int_{-\sigma_1}^{\sigma_2} \frac{z\varphi(z)dz}{\left|z^2 + H\right|^{\delta}} = J_1' + J_2',$$

where φ is any smooth function, σ_1 and σ_2 determined from the conditions $y(\sigma_1) = -\sigma,$ $y(\sigma_2) = \sigma.$ In the integrals J_1' and J_2' we make a linear change of variables $z = |H|^{1/2}t$. Then

$$J_1' = \frac{1}{\sqrt{3}|H|^{\delta - 1/2}}\int_{-\sigma_1|H|^{-1/2}}^{\sigma_2|H|^{-1/2}} \frac{dt}{\left|t^2 \pm 1\right|^{\delta}}. \tag{18}$$

It is easy to show that the integral (18) is uniformly bounded with respect to H, for $\frac{1}{2} < \delta < 1.$ From here we get

$$|J_1'| \le \frac{c}{H^{\delta - 1/2}}.$$

The integral J_2' is estimated as follows:

$$J_2' \le c\int_{-\sigma_1}^{\sigma_2} \frac{z\varphi(z)dz}{\left|z^2 + H\right|^{\delta}} = c\int_{-\sigma_1/H^{1/2}}^{\sigma_2/H^{1/2}} \frac{Ht\varphi(|H|^{1/2}t)dt}{\left|Ht^2 + H\right|^{\delta}} \le$$

$$\le \frac{c}{H^{\delta-1}}\int_{-\sigma_1/H^{1/2}}^{\sigma_2/H^{1/2}} \frac{t\varphi(|H|^{1/2}t)dt}{|t^2\pm 1|^{\delta}} \le \frac{c_1}{H^{\delta-1}}\int_2^{|H|^{-1/2}} t^{1-2\delta}dt + \frac{c_2}{H^{\delta-1}} =$$

$$= \frac{c_1 H^{1-\delta}}{2-2\delta} t^{2-2\delta}\Big|_2^{|H|^{-1/2}} + \frac{c_2}{H^{\delta-1}} = \frac{c_1 H^{1-\delta}}{2-2\delta}\left[H^{\delta-1} - 2^{2-2\delta}\right] + \frac{c_2}{H^{\delta-1}} \le C,$$

here the validity of the last inequality follows from the condition $\delta < 1$. Thus, $J_2^{'}$ is uniformly bounded. From the obtained estimates for $J_1^{'}$ and $J_2^{'}$, we get

$$\int_{-\sigma}^{\sigma} \frac{dy}{\left|y^2\left(y+\sqrt{3}\right)+H\right|^{\delta}} \le \frac{c}{|H|^{\delta-1/2}}.$$

Thus, under the conditions (11) and $\frac{1}{2} < \delta < 1$, we get the estimate

$$|J(p,q)| = \int_{-1}^{1} \frac{dx}{\left|x^3+px+q\right|^{\delta}} \le \frac{1}{|p|^{\frac{3\delta-1}{2}}} \frac{c}{\left|B-\frac{2}{3\sqrt{3}}\right|^{\delta-1/2}} = \frac{1}{|p|^{\frac{3\delta-1}{2}}} \times$$

$$\times \frac{c}{\left|\frac{q}{(-p)^{3/2}} - \frac{2}{3\sqrt{3}}\right|^{\delta-1/2}} \le \frac{c}{|p|^{1/4}} \frac{\left|q+\frac{2}{3\sqrt{3}}(-p)^{3/2}\right|^{\delta-1/2}}{|D|^{\delta-\frac{1}{2}}} \le \frac{c}{D^{\delta-\frac{1}{2}}\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{2-3\delta}{6}}}.$$

Now, suppose that the condition (12) holds true. In this case, we use the change of variables $x = |q|^{1/3} z$ for the integral (10). Then

$$|J(p,q)| = \frac{1}{|q|^{\delta-\frac{1}{3}}} \int_{-|q|^{-\frac{1}{3}}}^{|q|^{\frac{1}{3}}} \frac{dy}{\left|z^3+Az+sgn(q)\right|^{\delta}},$$

where $|A| = \left|\frac{p}{q^{2/3}}\right| \le \frac{3}{\sqrt[3]{4}}$.

If $A \in \left[-\frac{3}{\sqrt[3]{4}}+\varepsilon, \frac{3}{\sqrt[3]{4}}-\varepsilon\right]$ (where ε sufficiently small positive fixed number), then again the integral is uniformly bounded. The case $\left|A^2 - \frac{9}{\sqrt[3]{16}}\right| < \varepsilon$ is considered similarly to the case $\left|B^2 - \frac{4}{27}\right| < \varepsilon$. Theorem 2 is proved.

Proof of Theorem 1

For simplicity we suppose that $|a_0| = \max\{|a_i|, i = \overline{0,3}\}$, otherwise, by rotating the coordinate axes, we can reduce to this case. Note that, since D is the invariant of the group $SL(2,\mathbb{R})$, the integral and estimate do not depend on the choice of such change of variables. Moreover, the norm of the amplitude is uniformly bounded because the rotation group is a compact set. Let's represent the polynomial in the form

$$P_3(a,x,y) = a_0\left(x^3 + \frac{3a_1}{a_0}x^2y + \frac{3a_2}{a_0}xy^2 + \frac{a_3}{a_0}y^3\right).$$

We make change of variables as $x_1 := x + \frac{a_1}{a_0} y, \quad y_1 := y.$ Since $\left|\frac{a_1}{a_0}\right| \le 1,$ the Jacobian and the transformation norm are uniformly bounded. As a result,

$$P_3(a, x, y) = a_0\left(x_1^3 + px_1y_1^2 + qy_1^3\right) = a_0\Phi,$$

where $p = \frac{3a_0a_2 - 3a_1^2}{a_0^2}$ Pë $q = \frac{a_0^2a_3 + 2a_1^3 - 3a_0a_1a_2}{a_0^3}$. Note that, $|p| \le 6, \quad |q| \le 6.$ Hence the integral is represented as

$$J_{\alpha,\beta} := \int_{R^2} E_{\alpha,\beta}(ia_0\Phi)\psi(x_1, y_1)dx_1dy_1. \tag{19}$$

Passing to the polar coordinate system as $x_1 = r\cos\theta, y_1 = r\sin\theta,$ we get

$$J_{\alpha,\beta} \le \int_0^{2\pi}\int_0^{\infty} E_{\alpha,\beta}(ia_0r^3\phi(\cos\theta, \sin\theta))r\psi(r\cos\theta, r\sin\theta)drd\theta. \tag{20}$$

For the inner integral

$$J_{in} := \int_0^{\infty} E_{\alpha,\beta}(ia_0r^3\phi(\cos\theta, \sin\theta))r\psi(r\cos\theta, r\sin\theta)dr,$$

using Proposition 1, we get

$$|J_{in}| \le \int_0^{\infty} \frac{r|\psi(r\cos\theta, r\sin\theta)|dr}{1 + |a_0r^3\phi(\cos\theta, \sin\theta)|}.$$

Then changing the variable as $\rho = r^3|a_0\phi(\cos\theta, \sin\theta)|,$ we deduce

$$|J_{in}| \le \frac{c\|\psi\|_{C(R^2)}}{|a_0\phi(\cos\theta, \sin\theta)|^{\frac{2}{3}}} \int_0^{\infty} \frac{d\rho}{\rho^{\frac{1}{3}}(1+\rho)}.$$

Note that $\|\psi\|_{C(R^2)}$ is finite number, because ψ is a smooth function with compact support. As the last integral converges we have

$$|J_{in}| \le \frac{c\|\psi\|_{C(R^2)}}{|a_0\phi(\cos\theta, \sin\theta)|^{\frac{2}{3}}}.$$

Thus, the integral J has the estimate

$$|J_{\alpha,\beta}| \le \frac{c\|\psi\|_{C(R^2)}}{|a_0|^{\frac{2}{3}}} \int_0^{2\pi} \frac{d\theta}{|\phi(\cos\theta, \sin\theta)|^{\frac{2}{3}}}, \tag{21}$$

where $\phi = \cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta.$

Consider the last integral

$$J_2 := \int_0^{2\pi} \frac{d\theta}{|\phi(\cos\theta, \sin\theta)|^{\frac{2}{3}}}. \tag{22}$$

Lemma 1. *For the integral (22) the following estimate holds true*

$$|J_2| = \left|\int_0^{2\pi} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}}\right| \le \frac{c}{|D(\phi)|^{\frac{1}{6}}},$$

where $D(\phi) = \frac{p^3}{27} + \frac{q^2}{4}.$

Lemma 1 was proved in [9]. For convenience of the reader, we give a proof of Lemma 1 based on the Theorem 2.

Proof of Lemma 1. Note that

$$J_2 = 2\int_0^{\pi} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}} = 2(J_{20} + J_{21}),$$

where $J_{20} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}}$, $J_{21} = \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}}$.

First, we write

$$J_{20} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}} =$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}}.$$

Then we estimate

$$J_{20}^{'} = \int_0^{\frac{\pi}{4}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}}$$

making the change of variables $tg\theta = t$. We get

$$J_{20}^{'} = \int_0^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}}. \qquad (23)$$

Let $|t| \le \frac{1}{4}$ as $|p| \le 6, |q| \le 6$, then $|qt^3| \le \frac{6}{64}$, $|pt^2| \le \frac{6}{16}$. Hence,

$$\left|t^3 + pt^2\right| \le \frac{30}{64} \text{ and } \left|1 + qt^3 + pt^2\right| \ge 1 - \frac{30}{64} = \frac{34}{64}.$$

Then

$$\left|J_{20}^{'}\right| = \int_0^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}} = \int_0^{\frac{1}{4}} \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}} +$$

$$+ \int_{\frac{1}{4}}^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}} \le 1 + \int_{\frac{1}{4}}^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}}.$$

By changing the variables $x = \frac{1}{t}$ the last integral is reduced to the form

$$\int_{\frac{1}{4}}^1 \frac{dt}{|qt^3 + pt^2 + 1|^{\frac{2}{3}}} = \int_1^4 \frac{dx}{|x^3 + px + q|^{\frac{2}{3}}}. \qquad (24)$$

It is obvious that the identity:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{|\cos^3\theta + p\cos\theta\sin^2\theta + q\sin^3\theta|^{\frac{2}{3}}} = \int_0^1 \frac{dx}{|x^3 + px + q|^{\frac{2}{3}}}.$$

Thus, the problem reduces to estimating an integral of the form

$$\int_{N_1}^{N_2} \frac{dx}{|x^3+px+q|^{\frac{2}{3}}} \le \int_{-\infty}^{\infty} \frac{dx}{|x^3+px+q|^{\frac{2}{3}}}, \tag{25}$$

where N_1, N_2 are fixed numbers. Finally, the desired estimate for the integral (25) follows easily from Theorem 2 taking $\delta = \frac{2}{3}$. The estimate for the integral $J_{21}^{'}$ is similarly performed. Which completes the proof of Lemma 1.

Applying Lemma 1 for the integral (21), we obtain the required estimate. Theorem 1 is proved.

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