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ILMIY AXBOROTNOMA

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Obuna indeksi – yakka tartibdagi obunachilar uchun **- 5583,**
tashkilot, korxonalar uchun **- 5584**

MUNDARIJA

Faraz Mehmood, Soleev A. *New generalization of ostrowski-grüss like inequality on time scales*……..	2-9
Alimov A.A. *On the properties of solutions of a nonlinear filtration problem with a source and multiple nonlinearities*.....	10-17
Axatov A.R., Tojiyev M.R. *Fazo-vaqt tahlili va tayanch vektor mashinasi asosida yong'inni tezkor aniqlash algoritmi*……	18-25
Байжуманов А.А. *Проблемы минимизации специальных дизъюнктивных нормальных форм сложных конъюнкций высказываний систем нелинейных булевых уравнений*……	26-30
Djuraev A.S., Obidov A.E. *Pedagogik kadrlar malakasini oshirish uchun moslashuvchan elektron ta'lim tizimini modellashtirish*	31-39
Иванов А.К., Мухамедов А.К., Шаронов И.А., Эшбуриев Р.М. *Об измерении радиоактивности мукомольно-крупяной пищевой продукции*.....	40-46
Ishankulov A.F., Tursunova N.R., Ergasheva D.R., Khalilov K.F., Galyametdinov Y.G., Mukhamadiyev N.K. *Obtaining semiconductor CdS, CdSe AND CdSe/ZnS core/shell nanoparticles and their composites with polymers*……	47-51
Шодиев Д.С., Хайруллаев М., Махмудов Ш. *О задаче коши для бигармонического уравнения* ……	52-57
Рахматуллаев И.А., Ботиров Х.З., Курбонов А.К., Хайдаров Х.С., Ахмедова Ш.Ш, Абдирахматова М.П. *Способ регистрации слабых оптических сигналов с помощью резонаторных кювет и импульсно-периодических лазерных источников*……	58-65
Jo'rakulov I.T., Umrzoqov F.B. $Ge_2Sb_2Te_5$ *yupqa qatlam plyonkalarining solishtirma kontakt qarshiligini o'rganish*……	66-71
Ikromova D.I. *On the sharp estimates for the fourier transform of measures supported to surfaces related with* E_8 *type singularities*……	72-83
Arziqulov M.M., Shermuxamedov U.A. *Urganch shahri iqlimiy ko'rsatgichlarining o'zgarishi va atmosfera gazlari konsentratsiyasiga bog'liqligi*.....	84-88
Komilova N.K., Maxmudov B.X. *Geokriminogen vaziyat va uni tadqiq etishdagi yondashuvlar*……	89-94
Fayzullayev N.I., Tillayev S.U., Mamirzayev M.A. *Metandan nanouglerod va vodorod olish reaksiyasi tezligiga turli omillarning ta'siri*.....	95-101
Djabbarov O.R., Zaripova A.R., Sodiqova U.U. *Divergent ko'rinishdagi biologik populyatsiya sistemalarining avtomodel yechimlari asimptotikasi*....	102-106
Tursunova N.G., Musulmanov N.H., Fayzullaev N.I. *Mezog'ovakli uglerod sintezining maqbul sharoitini tanlash*……	107-116
Бурнашев В.Ф., Кайтаров З.Д. *Численное исследование многофазной фильтрации в деформируемой пористой среде с учетом внешнего воздействия*……	117-130
Turayeva B.I., Kutliyeva G.J., Zuhritdinova N.Y., Kamolova X.F., Maxkamov A. *Tokdan ajratib olingan bakteriyalarning uzum fitopatogen mikromitsetlariga nisbatan antifungal xususiyatlari*.....	131-135
Mardonov J.A. *Uch o`lchovli fazoda laplas maydoni uchun karleman formulasi*……	136-140

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ON THE SHARP ESTIMATES FOR THE FOURIER TRANSFORM OF MEASURES SUPPORTED TO SURFACES RELATED WITH E_8 TYPE SINGULARITIES

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Abstract. In this paper, we consider estimates for the Fourier transform of signed measures supported to smooth surfaces in the three-dimensional Euclidean space. These surfaces are given by the graph of a smooth function having a singularity of type E_8. The sharp L^p estimates for the Fourier transform of a measure are obtained.

Key words: Singularity, maximal functions, oscillatory integrals, Fourier transform.

О точных оценках преобразования Фурье мер, сосредоточенных на поверхностях, связанных с особенностями типа E_8

Аннотация. В данной работе мы рассматриваем оценки преобразования Фурье знакопеременных мер, сосредоточенных на гладких поверхностях трехмерного Евклидова пространства. Эти поверхности задаются графиком гладкой функции, имеющей особенность типа E_8. Получены точные L^p оценки преобразования Фурье мер.

Ключевые слова: Особенность, максимальные функции, осцилляторные интегралы, преобразование Фурье.

Tashuvchisi E_8 maxsuslikka ega bo’lgan, sirtlarda mujassamlashgan ishora almashinuvchi o’lchovlar Furye almashtirishining aniq baholari haqida

Annotatsiya. Ushbu maqolada biz uch o'lchovli Evklid fazosidagi silliq sirtlarda mujassamlashgan ishora almashinuvchi o'lchovlar Furye almashtirishining baholarini ko'rib chiqamiz. Bu sirtlar E_8 tipidagi maxsuslikka ega bo’lgan funksiyaning grafigi shaklida berilgan. O'lchovning Furye almashtirishi uchun aniq L^p baholar olingan.

Kalit so'zlar: Maxsuslik, maksimal funksiyalar, tebranuvchan integrallar, Furye almashtirishi.

1. Introduction and proposition of the problem. Let $S \subset \mathbb{R}^3$ be a smooth surface, and $\varphi \in C_0^\infty(S)$ be a smooth function with compact support. Consider the signed measure $d\mu(X) := \varphi(X)dS$, where dS is the induced Lebesgue measure on the surface S. Surely, if φ is a non-negative function, then we deal with the Borel’s surface-carried measure. The Fourier transform of the signed measure $d\mu$ is determined by the following integral:

$$\widehat{d\mu}(\xi) := \int_S e^{iX\xi}\, d\mu(X), \tag{1}$$

what corresponds to the Fourier transform of the distribution with compact support given by the signed measure $d\mu$ (see [4]), where $X \cdot \xi$ is the inner product of the vectors X and ξ.

In this paper we consider the problem: *Find the GLB (greatest lower bound) p_S of the following set*

$$\{p \in [1,\infty):\ it\ holds\ the\ inclusion:\ \widehat{d\mu} \in L^p(R^{n+1}) \quad for\ any \quad \varphi \in C_0^\infty(S)\},$$

where $L^p(\mathbb{R}^3)$ is the space of integrable functions with degree $p(1 \le p < \infty)$, of course, if $p = \infty$, then we are dealing with the space of essentially bounded functions. Similarly one can define $L^p_{loc}(\mathbb{R}^3)$.

The number p_S is called to be an exact integrability index of the Fourier transform of the signed measure $d\mu$ concentrated on the surface S. Such kind of problems had been considered by many researchers beginning from the classical papers by I.M. Vinogradov [6] and Hua Loo-keng [7].

In this paper, we consider the case of surfaces in $\mathbb{R}^3$ given as the graph of a function ϕ i.e. $S=\{x\in\mathbb{R}^3: x_3=\phi(x_1,x_2)\}$, the function ϕ has a singularity of type E_8 at the origin of $\mathbb{R}^2$, wherein the support of the density function φ belongs to a sufficiently small neighborhood of the origin.

Note that the well-known uniform estimates obtained by J.J. Duistermaat in the work [13] give the bound $p_S\le 3h(\phi)$, where $h(\phi)$ is the height of the function ϕ introduced by A.N. Varchenko [15]. In the case of singularities of type E_8 we have: $h(\phi)=15/8$. It should worth to say that the estimate $p_S\le 3h(\phi)=45/8$ is far from being a sharp. Actually, from the classical results it follows that the equality $p_S=3h(\phi)$ holds if and only if $h(\phi)=1$, i.e. ϕ has a non-degenerate critical point at the origin.

We will show that, there is a function having a singularity of type E_8 at the origin for which $p_S=3$ (see Theorem 1).

In order to prove our main result, we use the standard partition of unity trick which is often used in the theory of singular integrals (see [5], [14]). It turns out that the partition of unity method allows us to reduce the problem on behavior of oscillatory integrals (OI) to similar problems with a simpler phase, namely with phase, which is a deformation of a singularity of type A_k. At some cases estimates (OI) depending on the large and small parameters (the so-called combined estimates) obtained in [16] and [17] allow us to solve the problem integrability of the Fourier transform of measures.

However, we need to integrate obtained estimates over some small parameters and obtain uniform estimates with respect to other small parameters. Therefore, we often use a rougher "implicit" estimates in terms of the Randol's maximal functions (see [10], [11]). More precisely, "constants" on estimates of (OI) are determined by the sum of the convergent series. Moreover, we will show that the sum of the obtained series is integrable with some degree. These properties are enough to get upper bounds for the integrability exponent of the Fourier transform of measure.

The problem on the exact exponent of integrability of the Fourier transform of signed measures has several motives. One of the most famous motive associated with the method of trigonometric sums (see [2], also see [6], [7]). Another motive for the integrability problem is its connection with the Fourier restriction problem to sets of measure zero. Some results are known for the Fourier restriction to smooth surfaces, a more detailed discussion of known results are contained in the classical monograph by E.M. Stein [5] (also see [8] and [9]).

The paper organized as follows. In the next section 2 we formulate the main results of the paper. Also, we show the sharpness of the results. The next section 3 of the paper is devoted to estimate the one-dimensional oscillatory integrals depending on some small parameters. We obtain uniform estimates depending on small parameters. We consider estimates for the Randol's maximal functions in the section 4. Finally, in the last section 5 we give a proof of the main Theorem of the paper.

2. *Statement of the main results.* *In this paper, we will assume that the support of the density φ belongs to a sufficiently small neighborhood of the origin of $\mathbb{R}^3$ and also, it is assumed that the surface S, in a sufficiently small neighborhood of the origin, is given as the graph of a smooth function: $x_3=\phi(x_1,x_2)$, satisfying the conditions: $\phi(0)=0$ and $\nabla\phi(0)=0$.*

Denote by ϕ_k the homogeneous polynomial of order k obtained by Taylor's expansion of the function ϕ. It is well known that if $h(\phi)<2$ then the function ϕ is diffeomorphic equivalent to functions having simple singularities of Arnold's type [3]. But in the problem under consideration, we need to have the form of these functions with respect to a linear change of variables. Let's formulate statement on a normal form of functions having E type singularities with respect to linear change of variables (see [3] and also see [8]):

Proposition 2.1. *Let* ϕ *be a smooth function that satisfies the conditions:* $\frac{\partial^2\phi(0,0)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} = 0$, *for any* $(\alpha_1,\alpha_2)\in \mathsf{Z}_+^2$ *with* $\alpha_1+\alpha_2\le 2$ *(where* Z_+ *is the set of all non-negative integer numbers), and* $\phi_3(x_1,x_2)$ *is the cub of a linear function and also* $h(\phi)<2$. *Then there is a linear coordinate system of* R^2, *in which the function* ϕ *can be written as:*

$$\phi(x_1,x_2) = b(x_1,x_2)(x_2-\psi(x_1))^3 + x_2 b_1(x_1) + b_0(x_1), \tag{2}$$

where b, b_0, b_1 and ψ are smooth functions with $b(0,0)\neq 0$, and $\psi(0)=\psi'(0)=0$, as well as $b_1(x_1) = x_1^{k_1}\beta_1(x_1), b_0(x_1) = x_1^{k_0}\beta_0(x_1)$. Here β_1,β_0 are smooth functions.

Moreover, either

(i) $k_0 = 4, k_1\ge 4$, and also $\beta_0(0)\neq 0$ (singularity of type E_6);

(ii) $k_0\ge 5, k_1 = 3$ and also $\beta_1(0)\neq 0$ (singularity of type E_7);

(iii) $k_0 = 5, k_1\ge 5$, and also $\beta_0(0)\neq 0$ (singularity of type E_8).

The main results of the paper are contained in the following Theorem.

Theorem 1. *Let* $S\subset\mathsf{R}^3$ *be a smooth surface defined as the graph of a function* $x_3=\phi(x_1,x_2)$, *where* ϕ *is a smooth function that satisfies the conditions:*

(i) $\phi(0,0)=0,\quad \nabla\phi(0,0)=0$;

(ii) ϕ has a singularity of type E_8, while in the formula 2 the function ψ satisfies the condition: $|\psi''(0)|+|\psi'''(0)|\neq 0$.

Then there exists a neighborhood U of the origin such that for of all $\varphi\in C_0^\infty(U)$ the inclusion $\widehat{d\mu}\in L^{3+0}(\mathsf{R}^3)$ holds true, and if $\varphi(0)\neq 0$, then $\widehat{d\mu}\notin L^3(\mathsf{R}^3)$. So, in this case $p_S = 3$.

3. On estimates for one-dimensional oscillatory integrals. *In this section, we consider a one-dimensional oscillatory integral of the form:*

$$J(\lambda,s_1,\varepsilon,\delta) = \int_{\mathsf{R}} a(x_1,\varepsilon,\delta)e^{i\lambda\phi(x_1,s_1,\varepsilon,\delta)}dx_1, \tag{3}$$

where the amplitude function a is concentrated in a sufficiently small neighborhood of the origin.

Theorem 2. *Let* $J(\lambda,s_1,\varepsilon,\delta)$ *be one-dimensional oscillatory integral with phase*

$$\Phi(x_1,s_1,\varepsilon,\delta) := x_1^{n+1}B_n(x_1,\varepsilon,\delta) + s_1x_1 + \varepsilon x_1^{m+1}(1+\varphi(x_1,\varepsilon,\delta)), \tag{4}$$

where $1\le m<n$ and B_n,φ are smooth functions satisfying the conditions: $B_n(0,0,0)\neq 0$, $\varphi(0,0,0)=0$, and the amplitude support is located in a sufficiently small neighborhood of the origin. Then there exists a positive number $\varepsilon_0>0$ such that for $|\varepsilon|<\varepsilon_0, |\delta|<\varepsilon_0, |s_1|<\varepsilon_0$ for the integral $J(\lambda,s,\varepsilon,\delta)$ the following estimate holds:

$$|J(\lambda,s_1,\varepsilon,\delta)|\le \frac{\|a(\cdot,s_1,\varepsilon,\delta)\|_V\psi(s_1,\varepsilon,\delta)}{|\lambda|^{\frac{1}{2}}}. \tag{5}$$

Moreover, the function ψ satisfies the conditions:

$$\int_{|s_1|\le\varepsilon_0}(\psi(s_1,\varepsilon,\delta))^p ds_1 \le \begin{cases} C_p & for \quad 1\le p<\frac{2n}{n-1}; \\ C_p|\varepsilon|^{\frac{2n+(1-n)p}{2(n-m)}} & for \quad \frac{2n}{n-1}<p<\min\{\frac{2m}{m-1},4\}, \end{cases}$$

where C_p, C are positive numbers independent of ε,δ.

Remarks 3.1. *Note that the statement of the Theorem 2 follows from the general Theorem 1*

of the paper [1] *for the case* $p < \frac{2n}{n-1}$. *Actually in this case the statement of the Theorem 2 holds true for any phase function* $\Phi(x, s_1, \varepsilon, \delta) = \phi(x, \varepsilon, \delta) + s_1 x_1$, *where* $\phi(x, \varepsilon, \delta)$ *is a smooth perturbation of a function defined on a neighborhood of the origin of* R, *having* A_n *type singularity at the origin. However, in the proof of the Theorem 2 we essentially use dependence on* ε *of the phase function for the case when* $p > \frac{2n}{n-1}$.

First, we prove the following statement: There is a positive number Δ such that the following estimate:

$$| J(\lambda, s_1, \varepsilon, \delta) | \le \frac{C \| a(\cdot, s_1, \varepsilon, \delta) \|_V}{| s_1 |^{\frac{n-1}{2n}} | \lambda |^{\frac{1}{2}}}, \tag{6}$$

holds true, for the integral $J(\lambda, s_1, \varepsilon, \delta)$, whenever $| \varepsilon | \le \Delta | s_1 |^{\frac{n-m}{n}}$.

Simple computations show that the inequality

$$\int_{|\varepsilon| < \Delta |s_1|^{\frac{n-m}{n}}} \frac{ds_1}{| s_1 |^{\frac{(n-1)p}{2n}}} \le \frac{2\left(\frac{\varepsilon}{\Delta}\right)^{\frac{2n-(n-1)p}{2(n-m)}}}{\frac{(n-1)p}{2n} - 1}$$

holds, whenever $p > \frac{2n}{n-1}$.

Note that if $| \lambda | | s_1 |^{\frac{n+1}{n}} \le 1$ then the bound (6) easily follows from the classical Van der Corput's Lemma (see [2]). In what follows, we will assume that $| \lambda | | s_1 |^{\frac{n+1}{n}} > 1$. We show that actually, the inequality (6) holds whenever $| \varepsilon | \le \Delta | s_1 |^{\frac{n-m}{n}}$, provided Δ is a sufficiently small positive number.

Let's make a change of variables: $x_1 = | s_1 |^{\frac{1}{n}} y_1$ in the integral $J(\lambda, s_1, \varepsilon, \delta)$. Then we get:

$$J(\lambda, s_1, \varepsilon, \delta) = | s_1 |^{\frac{1}{n}} \int_{\mathsf{R}} e^{i\lambda |s_1|^{\frac{n+1}{n}} \Phi(y_1, s_1, \varepsilon, \delta)} a(| s_1 |^{\frac{1}{n}} y_1, s_1, \varepsilon, \delta) dy_1, \tag{7}$$

where

$$\Phi(y_1, s_1, \varepsilon, \delta) = y_1^{n+1} B_n(| s_1 |^{\frac{1}{n}} y_1, s_1, \varepsilon, \delta) + \frac{\varepsilon}{| s_1 |^{\frac{n-m}{n}}} y_1^{m+1} (1 + \varphi(| s_1 |^{\frac{1}{n}} y_1, s_1, \varepsilon, \delta) + sign(s_1) y_1.$$

It is easy to show that the set of critical points of the phase function is contained on some segment $[-M, M]$ (where M is a fixed positive number). We take a smooth function χ defined on R with compact support such that that $\chi(x) = 1$ for $| x | \le M + 1$. With this function, the integral (7) can be written as the sum of the following two integrals:

$$J(\lambda, s_1, \varepsilon, \delta) = J_1(\lambda, s_1, \varepsilon, \delta) + J_2(\lambda, s_1, \varepsilon, \delta),$$

where $J_1(\lambda, \sigma)$ and $J_2(\lambda, \sigma)$ are determined by the following equalities:

$$J_1(\lambda, s_1, \varepsilon, \delta) = | s_1 |^{\frac{1}{n}} \int_{\mathsf{R}} e^{i\lambda |s_1|^{\frac{n+1}{n}} \Phi(y_1, s_1, \varepsilon, \delta)} a(| s_1 |^{\frac{1}{n}} y_1, s_1, \varepsilon, \delta) \chi(y_1) dy_1, \tag{8}$$

and

$$J_2(\lambda,s_1,\varepsilon,\delta)=|s_1|^{\frac{1}{n}}\int_R e^{i\lambda|s_1|^{\frac{n+1}{n}}\Phi(y_1,s_1,\varepsilon,\delta)}a(|s_1|^{\frac{1}{n}}y_1,s_1,\varepsilon,\delta)(1-\chi(y_1))dy_1. \quad (9)$$

The phase function $\Phi(y_1,s_1,\varepsilon,\delta)$ has no critical points on the support of the amplitude of the oscillatory integral $J_2(\lambda,s_1,\varepsilon,\delta)$. Therefore, applying the van der Corput Lemma [2] we have an estimate of the form: (6).

Thus, it suffices to estimate the integral $J_1(\lambda,s_1,\varepsilon,\delta)$. The phase function of this integral can be considered as a small deformation of the following function: $y_1^{n+1}B_n(0,0,0,0)+sign(s_1)y_1$. Since $B_n(0,0,0,0)\neq 0$, there exists a positive number Δ such that the phase function has at worst non-degenerate critical points, whenever $|\varepsilon|\le\Delta|s_1|^{\frac{n-m}{n}}$, provided that other parameters are sufficiently small. Thus, similar to the first case, we have an estimate of the form (6).

Now suppose that $|\varepsilon|\ge\Delta|s_1|^{\frac{n-m}{n}}$. In this case, we make the change of variables $x_1=|\varepsilon|^{1/(n-m)}y_1$ in the integral $J_1(\lambda,s_1,\varepsilon,\delta)$ and obtain:

$$J_1(\lambda,s_1,\varepsilon,\delta)=:J(\lambda,\xi_1,\varepsilon,\delta)=|\varepsilon|^{\frac{1}{n-m}}\int_R e^{i\lambda|\varepsilon|^{\frac{n+1}{n-m}}\Phi_2(y_1,\xi_1,\varepsilon,\delta)}a(|\varepsilon|^{\frac{1}{n-m}}y_1,\xi_1,\varepsilon,\delta)dy_1, \quad (10)$$

where

$$\Phi_2(y_1,\xi_1,\varepsilon,\delta):=y_1^{n+1}B_n(|\varepsilon|^{\frac{1}{n-m}}y_1,|\varepsilon|^{\frac{n}{n-m}}\xi_1,\varepsilon,\delta)+$$
$$sign(\varepsilon)y_1^{m+1}(1+|\varphi(|\varepsilon|^{\frac{1}{n-m}}y_1,|\varepsilon|^{\frac{n}{n-m}}\xi_1,\varepsilon,\delta)+\xi_1 y_1,$$

here we used notation $\xi_1:=\dfrac{s_1}{|\varepsilon|^{\frac{n}{n-m}}}$. Note that by our assumption $|\xi_1|\le\Delta^{\frac{n}{m-n}}$.

It is easy to show that the set of critical points of the phase function is contained on some closed interval $[-N,N]$ (where N is a fixed positive number). Again with the help of a smooth functions χ with compact support such that $\chi(x)=1$ for $|x|\le N+1$ the integral (7) can be written as the sum of two integrals:

$$J(\lambda,\xi_1,\varepsilon,\delta)=J_1(\lambda,\xi_1,\varepsilon,\delta)+J_2(\lambda,\xi_1,\varepsilon,\delta),$$

where $J_1(\lambda,\xi_1,\varepsilon,\delta)$ and $J_2(\lambda,\xi_1,\varepsilon,\delta)$ are determined by the following equalities:

$$J_1(\lambda,\xi_1,\varepsilon,\delta):=|\varepsilon|^{\frac{1}{n-m}}\int_R e^{i\lambda|\varepsilon|^{\frac{n+1}{n-m}}\Phi_2(y_1,\xi_1,\varepsilon,\delta)}a(|\varepsilon|^{\frac{1}{n-m}}y_1,s_1,\varepsilon,\delta)\chi(y_1)dy_1, \quad (11)$$

and

$$J_2(\lambda,\xi_1,\varepsilon,\delta):=|\varepsilon|^{\frac{1}{n-m}}\int_R e^{i\lambda|\varepsilon|^{\frac{n+1}{n-m}}\Phi_2(y_1,\xi_1,\varepsilon,\delta)}a(|\varepsilon|^{\frac{1}{n-m}}y_1,s_1,\varepsilon,\delta)(1-\chi(y_1))dy_1. \quad (12)$$

The phase function $\Phi_2(y_1,\xi_1,\varepsilon,\delta)$ has no critical points on the support of the amplitude of the oscillatory integral $J_2(\lambda,\xi_1,\varepsilon,\delta)$. Therefore, similarly to the first case, we have the estimate of the following type:

$$|J_2(\lambda,\xi_1,\varepsilon,\delta)|\le\frac{C\|a(\cdot,s_1,\varepsilon,\delta)\|_V\chi_{|s_1|\le|\Delta\varepsilon|^{\frac{n}{n-m}}}(s_1)}{|\varepsilon|^{\frac{n-1}{2(n-m)}}|\lambda|^{\frac{1}{2}}}. \quad (13)$$

Now, we consider an estimate for the integral $J_1(\lambda,\xi_1,\varepsilon,\delta)$. First suppose that $m\le 2$. Then the

function $\Phi_2(y_1,\xi_1,\varepsilon,\delta)$ is R_+-versal deformation of a singularity of type $A_k(k\le 2)$ in small neighborhoods of critical points (see [3] also see [8]). Indeed, in this case Φ_2 is a deformation of the following function

$$\tilde{\Phi}_2(y_1) := y_1^{n+1}B_n(0,0,0,0) + sgn(\varepsilon)y_1^{m+1} + cy_1,$$

where c is a fixed real number. If $y_1 = y_1^0 \neq 0$ is a critical point of this function, i.e. $\tilde{\Phi}_2'(0) = 0$ then $|\tilde{\Phi}_2''(0)| + |\tilde{\Phi}_2'''(0)| \neq 0$. Then the desired estimate follows from the more general results of the work [12] (also see [16]). In particular, if $m = 2$ we have at worst A_2 (Airy type) singularities.

Suppose $m > 2$. Then all nonzero critical points (in particular, if $c \neq 0$ then all critical points of the function $\tilde{\Phi}_2$) of the phase function is the R_+-versal deformation of the singularity of type $A_k(k\le 2)$ in small neighborhoods of critical points. Consequently, the desired estimate for the integrals whose amplitude concentrated in small neighborhoods of these non-zero critical points follows from the general results of the work [12].

Lemma 3.2. *Let $\xi = \xi_1^0$ be a fixed point of the segment $[-\Delta^{\frac{n}{m-n}}, \Delta^{\frac{n}{m-n}}]$ and $y_1 = y_1^0$ be a critical point of the function $\tilde{\Phi}_2(y_1)$. There is a neighborhood $U\times V \subset \mathsf{R}\times\mathsf{R}^3$ the point $(y_1^0,0,0,\xi_1^0)$ and mapping $X: U_0\times V_0 \to U\times V$ (where $U_0\times V_0 \subset \mathsf{R}\times\mathsf{R}^3$ is a sufficiently small neighborhood of the point $(0,0,0,\xi_1^0)$) that satisfies the conditions:*

$$X(0,0,0,\xi_1^0) = y_1^0, \frac{\partial X(0,0,0,\xi_1^0)}{\partial y_1} \neq 0$$

as well as

$$\Phi_2(X(y_1,\varepsilon,\delta,\xi_1),\xi_1,\varepsilon,\delta) = y_1^{\nu+1} + \sigma_1(\varepsilon,\delta,\xi_1)y_1^{\nu-1} + \ldots + \sigma_{\nu-1}(\varepsilon,\delta,\xi_1)y_1 + \sigma_\nu(\varepsilon,\delta,\xi_1),$$

where $\sigma_1,\ldots,\sigma_\nu$ are smooth functions with $\sigma_j(0,0,\xi_1^0) = 0$ for $j = 1,\ldots,\nu-1$, and in the case $2 \le \nu \le m$ we have the relation:

$$\frac{\partial\sigma_{\nu-1}(0,0,\xi_1^0)}{\partial\xi_1} \neq 0.$$

The only difference between the Lemma 3.2 and the general Moser theorem [3] is that the derivative of the function $\sigma_{\nu-1}$ with respect to the variable ξ_1 is nonzero for sufficiently small values of the parameters ε,δ. See [1] for a proof of the Lemma 3.2.

It should be noted that $\nu \le m$. If $\nu = 1$, then the phase function has a nondegenerate critical point and the assertion of the Lemma 3.2 is a consequence of the classical Morse Lemma [18]. In the case when $\xi_1^0 = 0$ and $y_1^0 = 0$ the phase function $\tilde{\Phi}_2$ has a singularity of type A_m. Otherwise if $\xi_1^0 = 0$ and $y_1^0 \neq 0$, then it is easy to see that the phase function has a non-degenerate critical point.

If $|\xi_1 - \xi_1^0| < \delta$, (where δ is a sufficiently small number) then Φ_2 can be considered as a small smooth perturbation of the function

$$\tilde{\Phi}_2(y_1) = y_1^{n+1}B_n(0,0,0,0) \pm y_1^{m+1} + \xi_1^0 y_1,$$

here $\xi_1^0 \in [-\Delta^{\frac{n}{m-n}}, \Delta^{\frac{n}{m-n}}]$ is a fixed number. If $\xi_1^0 \neq 0$ and $y = y_1^0$ is a critical point of the function $\tilde{\Phi}_2$, then $y_1^0 \neq 0$, and multiplicity of the critical point is at most two. Moreover, if $m = 2$, then the phase function has a singularity of type A_2 at the origin, whenever $\xi_1^0 = 0$.

In other words, the phase function satisfies the conditions of the Lemma 2 with $n = m$, and formally we have: $\varepsilon = 0$.

Let $\chi_0 \in C_0^\infty(U_0)$ be a smooth function, where U_0 is a sufficiently small neighborhood of zero. Consider the integral

$$J_{1\chi_0}(\lambda,\sigma):=|\varepsilon|^{\frac{1}{n-m}}\int_{\mathbb{R}}e^{i\lambda|\varepsilon|^{\frac{n+1}{n-m}}\Phi(y_1,s_1,\varepsilon,\delta)}a(|\varepsilon|^{\frac{1}{n-m}}y_1,s_1,\varepsilon,\delta)\chi_0(y_1)dy_1.$$

Due to results of the paper [1] there exists $\psi_0(\varepsilon,\delta,\cdot)\in L^{\frac{2m}{m-1}-0}([-\Delta^{\frac{n}{m-n}},\Delta^{\frac{n}{m-n}}])$ such that for integral $J_{1\chi_0}(\lambda,\sigma)$ the following estimate :

$$|J_{1\chi_0}(\lambda,s_1,\varepsilon,\delta)|\le\frac{\|a(\cdot,s_1,\varepsilon,\delta)\|_V\,\chi_{|s_1|\le|\Delta\varepsilon|^{\frac{n}{n-m}}}(s_1)\psi_0\left(\delta,\varepsilon,\frac{s_1}{|\varepsilon|^{\frac{n}{n-m}}}\right)}{|\varepsilon|^{\frac{n-1}{2(n-m)}}|\lambda|^{\frac{1}{2}}} \tag{14}$$

holds true. Moreover, the function ψ_0 satisfies the condition:

$$\int_{[-\Delta^{\frac{n}{m-n}},\Delta^{\frac{n}{m-n}}]}(\psi_0(\varepsilon,\delta,\xi_1))^p\,d\xi_1\le C_p, \tag{15}$$

for any $1\le p<\frac{2m}{m-1}$, where the constant C_p does not depend on ε,δ [1].

Summing up the estimates we have the inequality (5) with the function:

$$\psi(\delta,\varepsilon,s_1):=\frac{\chi_{|s_1|\le|\Delta\varepsilon|^{\frac{n}{n-m}}}(s_1)\tilde{\psi}_0\left(\delta,\varepsilon,\frac{s_1}{|\varepsilon|^{\frac{n}{n-m}}}\right)}{|\varepsilon|^{\frac{n-1}{2(n-m)}}}+\frac{\chi_{|\varepsilon|\le\Delta|s_1|^{\frac{n-m}{n}}}(s_1)}{|s_1|^{\frac{n-1}{2n}}},$$

where $\tilde{\psi}_0$ is a function for which the estimate (15) holds true.

Obviously, this function satisfies the assertions of the Theorem 2.

4.Integrability of the Randol's maximal function associated to the oscillatory integrals.

In this section we define Randol's type maximal function associated with the Fourier transform of the signed measure $\widehat{d\mu}$ by the following relation:

$$M(\omega)=\sup_{r>0}r^{\frac{n}{2}}\left|\widehat{d\mu}(r\omega)\right|, \tag{16}$$

where $\omega\in S^2$, here S^2 is the unit sphere centered at the origin. It corresponds to the classical Randol's maximal function [10] (also see [11]).

Also, we introduce the local Randol's maximal functions associated to oscillatory integrals. Let $J(\lambda,s)$ be an oscillatory integral of the following form:

$$J(\lambda,s):=\int_{\mathbb{R}^2}a(x,s)e^{i\lambda\Phi(x,s)}dx, \tag{17}$$

where $a\in C_0^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ and the phase function $\Phi(x,s)$ is defined by the formula:

$$\Phi(x,s):=\phi(x_1,x_2)+s_1x_1+s_2x_2. \tag{18}$$

The next function is called to be a local Randol's maximal function (see [10]):

$$M_{loc}(s):=\sup_{|\lambda|>0}|\lambda J(\lambda,s)|. \tag{19}$$

We will prove the local integrability of this function.

Lemma 4.1. *Let ϕ have a singularity of type E_8 at the origin and also the support a is in a sufficiently small neighborhood of zero. Then the following statements are hold:*

(i) If the function ψ satisfies the conditions $\psi''(0)\ne0$, then the local Randol's maximal function M_{loc} belongs to $L_{loc}^{\frac{28}{9}-0}(\mathbb{R}^2)$;

(ii) if $\psi'''(0)\neq 0$, then the local Randol's maximal function M_{loc} belongs to $L^{3-0}_{loc}(\mathbb{R}^2)$.

Analogical result holds true for the Randol's maximal function associated to the Fourier transform of signed measures supported to corresponding surfaces.

Corollary 4.2. *Let S be a surface given by the graph of the function ϕ having a singularity of type E_8 at the origin and also the support φ is in a sufficiently small neighborhood of zero. Then the following statements are hold:*

(i) If the function ψ satisfies the conditions $\psi''(0)\neq 0$, then the Randol's maximal function M associated to the Fourier transform of measures supported to S belongs to $L^{\frac{28}{9}-0}(S^2)$;

(ii) if $\psi'''(0)\neq 0$, then the Randol's maximal function M associated to the Fourier transform of measures supported to S belongs to $L^{3-0}(S^2)$.

Now, we move to a proof of the Lemma 4.1.

Let $J(\lambda,s)$ be an oscillatory integral of the following form:

$$J(\lambda,s):=\int_{\mathbb{R}^2}a(x,s)e^{i\lambda\Phi(x,s)}dx, \tag{20}$$

where

$$\Phi(x,s):=b(x_1,x_2)x_2^3+x_2x_1^5\beta_1(x_1)+x_1^5\beta_0(x_1)+s_1x_1+s_2x_2+s_2x_1^2\omega(x_1).$$

Let $\kappa_1=\frac{1}{5},\kappa_2=1/3$ and also for a positive number r, δ_r denotes dilatation, defined by the following formula: $\delta_r(x):=(r^{\kappa_1}x_1,r^{\kappa_2}x_2)$. Let us introduce the $\kappa-$ quasidistance as follows:

$$\rho_\kappa(x_1,x_2):=(x_1^{10}+x_2^6)^{\frac{1}{2}}.$$

Let β,χ_ν be smooth functions defined by equalities:

$$\beta(x)=\begin{cases}1 & for\ \ |x|\leq 1\\ 0 & for\ \ |\delta_{2^{-1}}(x)|\geq 1\end{cases}$$

$$\chi_\nu(x)=\beta(\delta_{2^{-\nu}}(x))-\beta(\delta_{2^{-\nu-1}}(x)).$$

It is easy to see that the following partition of unity relation

$$\sum_{\nu=-\infty}^{\infty}\chi_\nu(x)=1,\ \ for\ any\ \ x\neq 0 \tag{21}$$

holds true.

We assume that the support of the amplitude of oscillatory integral $J(\lambda,s)$ is in the set $\{\rho_\kappa(x)<2^{-\nu_0-1}\}\times V$, where $V\subset\mathbb{R}^2$ is a sufficiently small neighborhood of zero and ν_0 is a sufficiently large number. Then we use the dyadic partition of unity and obtain:

$$J(\lambda,s)=\sum_{\nu=\nu_0}^{\infty}J_\nu(\lambda,s).$$

In the integral $J_\nu(\lambda,s)$ we make a change of variables given by the scaling: $\delta_{2^\nu}x\to x$ and get:

$$J_\nu(\lambda,s):=2^{-\nu|\kappa|}\int_{\mathbb{R}^2}a(\delta_{2^{-\nu}}(x),s)\chi_1(x)e^{i\lambda 2^{-\nu}\Phi_\nu(x,\sigma)}dx,$$

where

$$|\kappa|:=\kappa_1+\kappa_2,\Phi_\nu(x,\sigma):=\phi_\nu(x_1,x_2)+\sigma_1x_1+\sigma_2x_2+\sigma_2 2^{(\kappa_2-(m+1)\kappa_1)\nu}x_1^{m+1}\omega(2^{-\kappa_2\nu}x_1),$$

here

$$\phi_\nu(x_1,x_2):=b(\delta_{2^{-\nu}}(x))(x_2^3+2^{(1-\kappa_2-5\kappa_1)\nu)}x_2x_1^5\beta_1(2^{-\kappa_1\nu}x_1)+x_1^5\beta_0(2^{-\kappa_1\nu}x_1)$$

and $\sigma_j := 2^{(1-\kappa_j)\nu} s_j\,(j=1,2)$.

Note that the support of the amplitude is in the annulus $D := \{x \in \mathbb{R}^2 : \frac{1}{2} \leq |x| \leq 2\}$ and $\nu \geq \nu_0$ where ν_0 is a sufficiently large natural number. Therefore, we can assume that the function $\phi_\nu(x_1, x_2)$ is a small and smooth deformation of the function $b(0,0)x_2^3 + x_1^5\beta_0(0)$. That's why there exist positive numbers C_1 and ν_0 such that for $|\sigma| > C_1$ and $\nu \geq \nu_0$ the phase function has no critical points. Therefore we can use integration by parts arguments and obtain that the following estimate:

$$|J_\nu(\lambda, s)| \leq \frac{C2^{(1-|\kappa|)\nu}\|a(\cdot,s)\|_{C^1}}{|\sigma|\,|\lambda|}. \tag{22}$$

The estimate (22) is better than wanted.

Also, there is a positive number $\varepsilon > 0$ such that that the estimate

$$|J_\nu(\lambda, s)| \leq \frac{C2^{(1-|\kappa|)\nu}\|a(\cdot,s)\|_{C^1}}{|\lambda|} \tag{23}$$

holds, for $|\sigma| < \varepsilon$.

Now, we consider an estimate for the integral $J_\nu(\lambda, s)$ under the condition $\sigma \in \Sigma := \{\varepsilon \leq |\sigma| \leq C_1\}$. In that case, standard finite covering arguments of the compact set Σ are used.

We fix $\sigma^0 \in \Sigma$. As in the previous case, if $\sigma_1^0 \neq 0 \neq \sigma_2^0$, the phase function has only non-degenerate critical points and they are separated from each other. Therefore, in this case we can use Morse Lemma and have the estimate (23) for the integral $J_\nu(\lambda, s)$, whenever $|\sigma - \sigma^0| \leq \delta$ (where δ is a sufficiently small positive number).

If $\sigma^0 \in \Sigma$ and $\sigma_1^0 \neq 0$, then $\sigma_2^0 = 0$ and the phase function is the $\mathbb{R}_+ -$versal deformation of singularities of type A_2. In this case the required bound follows from more general result proved in the paper [12].

Finally, let $\sigma^0 \in \Sigma$ and $\sigma_1^0 = 0$, then $\sigma_2^0 \neq 0$. Then the phase function $\Phi_\nu(x, \sigma)$ can be considered as a small perturbation of A_4 type singularity. So, we can use stationary phase method in x_2 variable. Note that the phase function has at worst two different non-degenerate critical points with respect to x_2: $x_2^\pm = x_2^\pm(2^{-\kappa_1\nu}x_1, \sigma_2)$ with $x_2^\pm(0, \sigma_2^0) \neq 0$. If it has no critical points in x_2 then we can use integration by parts arguments and have an estimate better than wanted. Further, we assume that the phase function has two non-degenerate critical points in x_2 variable. Moreover, it is easy to see that the function $\Phi_\nu(x_1, x_2^\pm(2^{-\kappa_1\nu}x_1, \sigma_2), \sigma)$ satisfies the conditions of the Theorem 2 with $\varepsilon := 2^{-\frac{\nu}{15}} = 2^{(\kappa_2 - 2\kappa_1)\nu}$ for $m = 1$. If $m = 2$ then $\varepsilon := 2^{-\frac{4\nu}{15}} = 2^{(\kappa_2 - 3\kappa_1)\nu}$, which corresponds to the case $\psi(0) = \psi'(0) = \psi''(0) = 0$ and also $\psi'''(0) \neq 0$.

Thus, we can apply the Theorem 2 and get estimate (5) with the function ψ_ν, and the function ψ_ν satisfies the following conditions:

$$\int_{|\sigma_1| \leq \varepsilon_0} (\psi_\nu(\sigma_1, \sigma_2))^p\, d\sigma_1 \leq C_p 2^{(\kappa_2 - (m+1)\kappa_1)\frac{8-3p}{2(4-m)}\nu} \quad for \quad \frac{8}{3} < p < 4,$$

here C_p is a positive number independent of $\nu > \nu_0, \sigma_2 \in \{|\sigma_2 - \sigma_2^0| < \varepsilon_0\}$.

Note that for $\frac{8}{3} < p$ we have: $(\kappa_2 - (m+1)\kappa_1)\frac{8-3p}{2(4-m)} > 0$ for $m = 1,2$. Therefore, we

have estimate (5) with the function $\tilde{\Psi}_\nu$, and for the function $\tilde{\Psi}_\nu$ the following inequality

$$\int_\Sigma (\tilde{\Psi}_\nu(\sigma_1,\sigma_2))^p d\sigma \le C_p 2^{(\kappa_2-(m+1)\kappa_1)\frac{8-3p}{2(4-m)}\nu} \quad for \quad \frac{8}{3} < p < 4,$$

holds true.

Let $V \subset \mathbb{R}^2$ be a sufficiently small neighborhood of zero and $1 \le p < 4$ is a fixed number. Then we have:

$$\int_{V_\nu} (\tilde{\Psi}_\nu(2^{(1-\kappa_1)\nu}s_1, 2^{(1-\kappa_2)\nu}s_2))^p ds_1 ds_2 = 2^{(|\kappa|-2)\nu}\int_\Sigma (\tilde{\Psi}_\nu(\sigma_1,\sigma_2))^p d\sigma_1 d\sigma_2 \le$$

$$C_p 2^{(|\kappa|-2+(\kappa_2-(m+1)\kappa_1)\frac{8-3p}{2(4-m)})\nu},$$

where

$$V_\nu := \{(s_1,s_2) : \varepsilon < |2^{(1-\kappa_1)\nu}s_1| + |2^{(1-\kappa_2)\nu}s_2| < C\}.$$

Finally, let's define:

$$\Psi_\nu(s) := 2^{(1-|\kappa|)\nu}(C\chi_{|(2^{(1-\kappa_1)\nu}s_1, 2^{(1-\kappa_2)\nu}s_2)|<\varepsilon}(s)\|a(\cdot,s)\|_{C^1} +$$

$$\frac{C\chi_{|(2^{(1-\kappa_1)\nu}s_1, 2^{(1-\kappa_2)\nu}s_2)|>C_1}(s)\|a(\cdot,s)\|_{C^1}}{|\sigma|} + \chi_{V_\nu}(s)\psi_\nu(2^{(1-\kappa_1)\nu}s_1, 2^{(1-\kappa_2)\nu}s_2)).$$

It is obvious that for $2 < p < 4$ the following relation holds true:

$$\int_V (\Psi_\nu(s_1,s_2))^p ds_1 ds_2 = 2^{(1-|\kappa|)\nu p+(|\kappa|-2)\nu}\int_{\mathbb{R}^2} (\psi_\nu(\sigma_1,\sigma_2))^p d\sigma_1 d\sigma_2 \le$$

$$C_p 2^{(1-|\kappa|)\nu p+(|\kappa|-2+((\kappa_2-(m+1)\kappa_1)\frac{8-3p}{2(4-m)})\nu}.$$

Since $|\kappa| = \frac{8}{15}$, the next series

$$\Psi(s_1,s_2) := \sum_{\nu=\nu_0}^{\infty} \Psi_\nu(s_1,s_2)$$

converge in the space $L^p(V)$ for $2 < p < \frac{28}{9}$ whenever $\psi''(0) \neq 0$. However, in the case $\psi'(0) = \psi'(0) = \psi''(0) = 0$ and also $\psi'''(0) \neq 0$ the series converge for any $2 < p < 3$. Moreover, V has bounded measure, and therefore $\Psi \in L^{\frac{28}{9}-0}(V)$ or $\Psi \in L^{3-0}(V)$ for $m = 1$ or $m = 2$ respectively.

What completes a proof of the Lemma 4.1 for the case when $\psi(0) = \psi'(0) = 0$ and $|\psi''(0)| + |\psi'''(0)| \neq 0$.

5.Proof of the main Theorem 1

Actually, a proof of the Theorem 1 follows from the Lemma 4.1 and Corollary 4.1. Indeed, since $\widehat{d\mu}$ is a bounded function it is enough to consider estimate of the integral over the set $\{|\xi| > 1\}$. By the Lemma 4.1 and Corollary 4.2 the following estimate holds true:

$$\widehat{d\mu}(r\omega) \le M(\omega)r^{-1}. \tag{24}$$

Moreover $M \in L^{\frac{28}{9}-0}(S^2)$ for the case when $\psi''(0) \neq 0$. Let $3 < p < \frac{28}{9}$ be a fixed number. Then by using polar coordinates system we have

$$\int_{\xi \ge 1} \left|\widehat{d\mu}(\xi)\right|^p d\xi \le \int_1^\infty \frac{dr}{r^{p-2}} \int_{S^2} M^p(\omega) d\omega \le +\infty.$$

Now, suppose $\psi(0)=\psi'(0)=\psi''(0)=0$ and also $\psi'''(0)\neq 0$. Then we use the following uniform estimate proved by J.J. Duistermaat [13]:

$$\left|\widehat{d\mu}(\xi)\right| \le \frac{C}{|\xi|^{\frac{8}{15}}}. \tag{25}$$

Note that if $p > \frac{45}{8}$ then due to the estimate (25) we have the relation $\widehat{d\mu} \in L^p(\mathbb{R}^3)$.

Suppose $3 < p < \frac{45}{8}$. Then we interpolate the estimates (24) and (25) and have

$$\left|\widehat{d\mu}(\xi)\right| \le \frac{CM^{\alpha}}{|\xi|^{\frac{8(1-\alpha)}{15}+\alpha}}, \tag{26}$$

where $0<\alpha<1$ is a fixed number.

We choose the number α from the inequalities:

$$0 < \frac{8}{7}\left(\frac{45}{8p}-1\right) < \alpha < 1.$$

Then we obtain:

$$3 < p\left(\alpha+\frac{8(1-\alpha)}{15}\right).$$

Finally, again by using polar coordinates system we have:

$$\int_{|\xi|\ge 1}\left|\widehat{d\mu}(\xi)\right|^p d\xi \le \int_1^{\infty}\frac{dr}{r^{\frac{8p(1-\alpha)}{15}+p\alpha-2}}\int_{S^2}M^{p\alpha}(\omega)d\omega \le +\infty.$$

Note that for $p\le 3$ we have $\widehat{d\mu}\notin L^3(\mathbb{R}^3)$ (see [1], Proposition 1). Theorem 1 is proved.

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