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NEW GENERALIZATION OF OSTROWSKI-GRÜSS LIKE INEQUALITY ON TIME SCALES

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Abstract. In the present paper we obtain the generalized inequality of Ostrowski-Grüss like on time scales using three steps kernel and recapture several established results of different authors of different papers and thus unify corresponding discrete and continuous versions. Further that we apply our obtained consequence to the case of quantum calculus.

Key words: Ostrowski inequality, Grüss inequality, Ostrowski-Grüss inequality, Time scales, Quantum calculus.

Vaqt shkalasida Ostrovski-Grüss ko'rinishdagi tengsizlikning yangi umumiylashmasi

Annotatsiya. Ushbu maqolada biz uch bosqichli yadrodan foydalangan holda vaqt shkalasida Ostrovski-Grüssning umumlashtirilgan tengsizligini isbotlaymiz va shu mavzudagi turli maqolalarining olingan natijalarini xususiy holda keltirib chiqaramiz hamda shu bilan tegishli diskret va uzluksiz hollarni birlashtiramiz. Bundan tashqari, biz olingan natijani kvant hisobi holatiga qo'llaymiz.

Kalit so'zlar: Ostrovski tengsizligi, Grüss tengsizligi, Ostrowski-Grüss tengsizligi, vaqt shkalasi, kvant hisobi.

Новое обобщение неравенства типа Островского-Грусса на часовых шкалах

Аннотация. В настоящей статье мы получаем обобщенное неравенство Островского-Грюсса на часовых шкалах, используя трёхразовое ядро, и восстановим несколько установленных результатов разных авторов из разных статей и, таким образом, объединяем соответствующие дискретные и непрерывные версии. Далее мы применяем полученное нами следствие к случаю квантового исчисления.

Ключевые слова: Неравенство Островского, Неравенство Грюсса, Неравенство Островского-Грюсса, Шкалы времени, Квантовое исчисление.

1. **Introduction.** In 1997, by using of Grüss & Ostrowski inequalities, Dragomir & Wang have derived the integral inequality of Ostrowski-Grüss like, (see [8]) which is stated as below:

Theorem 1.1. *Suppose* $g: I \to \mathbb{R}$ *is differentiable mapping in the open interval* $I \subset R$; $k, j \in I$ *and* $k > j$ *. If for real constants* γ, Γ; $\gamma \leq \overrightarrow{g'(\theta)} \leq \Gamma$ *for all* $\theta \in [j, k]$. *Then following inequality holds:*

$$\left| g(\theta) - \frac{1}{k-j}\int_j^k g(\tau)d\tau - \frac{g(k)-g(j)}{k-j}\left(\theta - \frac{j+k}{2}\right)\right| \leq \frac{k-j}{4}(\Gamma - \gamma) \qquad (1.1)$$

for all $\theta \in [j, k]$.

The above inequality has a relationship between the Ostrowski [24] & Grüss inequality [23]. It may be used for estimation of error bounds for some especial means and for some numerical quadrature formulae. For other closed consequences on the similar inequalities, (see [7, 14, 16, 25]).

The main objective of the paper is to extend a generalization of Ostrowski-Grüss like inequality to an arbitrary time scales.

2. **Essentials of Time Scales.** The idea of time scales calculus' theory was initiated and introduced by Hilger (1988) in his PhD thesis [11] (supervised by Aulbach) in order to unify discrete & continuous analysis and to expend the discrete & continuous theories to cases "in between". Since

then, mathematical research in this field has exceeded more than thousand publications and a lot of applications to all fields of science such as: operations research, economics, physics, engineering, statistics, finance and biology [6]. Even The time scale calculus theory can be used to any branch of science in which dynamic processes are explained by discrete-time / continuous-time models. We prefer the researcher to the book [2] written by Bohner & Peterson about an introduction to the singled variable time scales calculus and its implementations.

In 2004, M. Bohner introduced the calculus of variations on time scales, "he used the delta derivative and delta integral [3], and it has since then been further developed by several different authors in several different publications (see [1, 10, 12, 13, 17, 18]). Many classical results of calculus of variations as necessary or sufficient conditions of optimality have been generalized to arbitrary time scale.

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers.

The most important examples of time scales are $\mathbb{R}, \mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^k \big| \, k \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma: \mathbb{T} \to \mathbb{T}$ by $\sigma(\theta) := \inf\{\tau \in \mathbb{T} | \, \tau > \theta\}$ for all $\theta \in T$, the *backward jump operator* $\rho: \mathbb{T} \to \mathbb{T}$ by $\rho(\theta) := \sup\{\tau \in \mathbb{T} | \, \tau < \theta\}$ for all $\theta \in \mathbb{T}$. If $\sigma(\theta) > \theta$, then we say that θ is right-scattered, while if $\rho(\theta) < \theta$ then we say that θ is left-scattered.

Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(\theta) > \theta$, the θ is called right-dense, and if $\rho(\theta) < \theta$ then θ is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

Definition 2.3. Let $\theta \in \mathbb{T}$, then two mappings $u, v : \mathbb{T} \to [0, +\infty)$ satisfying

$$u(\theta) := \sigma(\theta) - \theta, v(\theta) := \theta - \rho(\theta) \qquad (2.1)$$

are called the graininess functions.

We now introduce the set $\mathbb{T}^c$ which is derived from the time scales $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum θ, then $\mathbb{T}^c := \mathbb{T} - \{t\}$, otherwise $\mathbb{T}^c := \mathbb{T}$. Furthermore for a function $g : \mathbb{T} \to \mathbb{R}$, we define the function $g^\sigma : \mathbb{T} \to \mathbb{R}$ by $g^\sigma(\theta) = g(\sigma(\theta))$ for all $\theta \in \mathbb{T}$.

Definition 2.4. Let $g : \mathbb{T} \to \mathbb{R}$ be a function on time scales. Then for $\theta \in \mathbb{T}^c$, we define $g^\Delta(\theta)$ to be the number, if one exists, such that for all $0 < \varepsilon$ there is a neighborhood U of θ such that for all $\tau \in U$

$$\left|g^\sigma(\theta) - g(\tau) - g^\Delta(\theta)(\sigma(\theta) - \tau)\right| \leq \varepsilon|\sigma(\theta) - \tau|.$$

We say that g is Δ −differentiable on $\mathbb{T}^c$ provided $g^\Delta(\theta)$ exists for all $\theta \in \mathbb{T}^c$.

Definition 2.5. A function $g: \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) provided if it satisfies

(i) g is continuous at each right-dense point or maximal element of $\mathbb{T}$.

(ii) The left-sided limit $\lim_{\tau \to \theta-} g(\tau) = g(\theta-)$ exists at each left-dense point θ of $\mathbb{T}$.

Definition 2.6. A function $G: \mathbb{T} \to R$ is called a Δ −antiderivative of $g: \mathbb{T} \to R$ provided $G^\Delta(\theta) = g(\theta)$ holds for all $\theta \in \mathbb{T}^c$. Then the Δ −integral of g is defined by

$$\int_j^k g(\theta)\Delta\theta = G(k) - G(j).$$

Remark 2.7. It follows from Theorem 1.74 of [2] that every rd-continuous function has an anti-derivative.

Proposition 2.8. *Let g, h be rd-continuous, $j, k, l \in \mathbb{T}$ and $\alpha, \beta \in R$. Then*

(i) $\int_j^k [\alpha g(\theta) + \beta h(\theta)]\Delta\theta = \alpha \int_j^k g(\theta)\Delta\theta + \beta \int_j^k h(\theta)\Delta\theta$,

(ii) $\int_j^k g(\theta)\Delta\theta = -\int_j^k g(\theta)\Delta\theta$,

(iii) $\int_j^k g(\theta)\Delta\theta = \int_j^l g(\theta)\Delta\theta + \int_l^k g(\theta)\Delta\theta$,

(iv) $\int_j^k g(\theta)h^\Delta(\theta)\Delta\theta = (gh)(k) - (gh)(j) - \int_j^k g^\Delta(\theta)h(\sigma(\theta))\Delta\theta$,

(v) $\int_j^j g(\theta)\Delta\theta = 0$.

Definition 2.9. Let $f_c : \mathbb{T}^2 \to \mathbb{R}, c \in \mathbb{N}_0$ be defined by

$$f_0(\theta, \tau) = 1 \qquad for\ all\ \tau, \theta \in \mathbb{T}$$

and then recursively by

$$f_{c+1}(\theta,\tau)=\int_{\tau}^{\theta} f_c(s,\tau)\Delta s \qquad for\ all\ \tau,\theta\in\mathbb{T}.$$

The present paper is motivated by the following results: Grüss inequality on time scales and Ostrowski inequality on time scales by Bohner & Matthews and also generalized Grüss inequality, generalized Ostrowski inequality and generalized Ostrowski-Grüss like inequality on time scales by authors in different papers [19, 20, 21]".

In 2007, Bohner & Matthews proved the Grüss inequality on time scales in Theorem 3.1 of [4] which is stated as:

Theorem 2.10. *Let $j,k,\tau\in\mathbb{T}$, $g,h\in C_{rd}$ and $g,h:[j,k]\to\mathbb{R}$. Then for*

$$m_1\le g(\tau)\le M_1\,,\qquad m_2\le h(\tau)\le M_2 \tag{2.2}$$

we have

$$\left|\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)h^{\sigma}(\tau)\Delta\tau-\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau\cdot\frac{1}{k-j}\int_j^k h^{\sigma}(\tau)\Delta\tau\right| \le\frac{1}{4}(M_1-m_1)(M_2-m_2)\ . \tag{2.3}$$

In 2008, Bohner & Matthews also proved the Ostrowski inequality on time scales in Theorem 3.5 of [5] which is stated as:

Theorem 2.11. Let $j,k,\tau,\theta\in\mathbb{T}$, $j<k$ *and* $g:[j,k]\to\mathbb{R}$ *be differentiable. Then*

$$\left|g(\theta)-\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau\right|\le\frac{M}{k-j}\big(f_2(\theta,j)+f_2(\theta,k)\big), \tag{2.4}$$

where

$$M=\sup_{j<\theta<k}|g^{\sigma}(\theta)|.$$

This inequality is sharp in the sense that the right-hand side of (2.4) cannot be replaced by a smaller one.

In 2009, Liu & Ngô derived a inequality of Ostrowski-Grüss like on time scales and they obtained unified results about discrete & continuous versions and they also gave the result to the case of quantum calculus. Recently, Mehmood, Khan & Shaikh also derived a inequality of Ostrowski-Grüss like on time scales and they also obtained unified results about discrete & continuous versions. In the current paper we will derive a generalized inequality of Ostrowski-Grüss like on time scales by using Theorem 2.10 and we recapture the results of different papers [5, 9, 15, 20, 21, 22]. Moreover, We would use our established result to the case of quantum calculus.

3. **Generalisation of Ostrowski-Grüss Like Inequality on Time Scales.** To prove the our main Theorem 3.13, we need the following generalized Montgomery Identity.

Lemma 3.1. *(Generalized Montgomery Identity). Let $j,k,\tau,\theta\in\mathbb{T}, j<k$ and $g:[j,k]\to\mathbb{R}$ be differentiable and $\lambda\in[0,1]$. Then*

$$(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}=\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau +\frac{1}{k-j}\int_j^k p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau \tag{3.1}$$

where

$$p(\theta,\tau)=\begin{cases}\tau-\left(j+\lambda\dfrac{k-j}{2}\right), & j\le\tau<0,\\ \tau-\dfrac{j+k}{2}\ , & 0\le\tau<j+k-\theta,\\ \tau-\left(k-\lambda\dfrac{k-j}{2}\right)\ , & j+k-\theta\le\tau\le k.\end{cases}$$

Proof. Apply Preposition 2.8 (iv), we have

$$\int_j^{\theta}\left(\tau-\left(j+\lambda\frac{k-j}{2}\right)\right)g^{\Delta}(\tau)\Delta\tau=\left(\theta-\left(j+\lambda\frac{k-j}{2}\right)\right)g(\theta)+\lambda\frac{k-j}{2}g(j)-\int_j^{\theta}g^{\sigma}(\tau)\Delta\tau,$$

$$\int_{\theta}^{j+k-\theta}\left(\tau-\frac{j+k}{2}\right)g^{\Delta}(\tau)\Delta\tau=\left(\frac{j+k}{2}-\theta\right)g(j+k-\theta)-\left(\theta-\frac{j+k}{2}\right)g(\theta)-\int_{\theta}^{j+k-\theta}g^{\Delta}(\tau)\Delta\tau,$$

and

$$\int_{j+k-\theta}^{k}\left(\tau-\left(k-\lambda\frac{k-j}{2}\right)\right)g^{\Delta}(\tau)\Delta\tau$$
$$=-\left(j-\theta+\lambda\frac{k-j}{2}\right)g(j+k-\theta)+\lambda\frac{k-j}{2}g(k)-\int_{j+k-\theta}^{k}g^{\sigma}(\tau)\Delta\tau.$$

Therefore

$$\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau+\frac{1}{k-j}\int_j^k p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau$$
$$=\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau+\frac{1}{k-j}\left[(k-j)(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)\right.$$
$$\left.+\lambda(k-j)\left(\frac{g(j)+g(k)}{2}\right)-\int_j^k g^{\sigma}(\tau)\Delta\tau\right],$$
$$=(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2},$$

i.e., (3.1) holds.

Remark 3.2. If we put $j+k-\theta=\theta$ in Lemma 3.1, we recapture the Montgomery Identity with parameter on time scale which are cited in Lemma 3.1 of [21] and Lemma 3.1 of [20].

Remark 3.3. If we put $j+k-\theta=\theta$ in Lemma 3.1, then put $\lambda=0$, we recapture the Montgomery Identity on time scale which are cited in Lemma 1 of [15] and Lemma 3.1 of [5].

Discrete Case:

Corollary 3.4. *We let* $\mathbb{T}=\mathbb{Z}$. *Let* $j=0,k=n,\tau=b,\theta=a$ *and* $g(c)=y_c$. *Then*

$$(1-\lambda)\left(\frac{y_a+y_{n-a}}{2}\right)+\lambda\left(\frac{y_0+y_n}{2}\right)=\frac{1}{n}\sum_{b=1}^{n}y_b+\frac{1}{n}\sum_{b=0}^{n-1}p(a,b)\Delta y_b\,,$$

where

$$p(a,0)=0$$

and

$$p(a,b)=\begin{cases}b-\dfrac{\lambda_n}{2}\,, & 0\le b\le a-1,\\ b-\dfrac{n}{2}\,, & a\le b\le n-a-1,\\ b-n\left(1-\dfrac{\lambda}{2}\right), & n-a\le b\le n-1.\end{cases}$$

as we just need $1\le a\le n$ *and* $0\ \le\ b\ \le\ n\ -1$.

Remark 3.5. If we put $n-a=a$ in Corollary 3.4, we recapture the discrete Montgomery Identity with parameter which are cited in Corollary 3.3 of [21] and Corollary 3.3 of [20].

Remark 3.6. If we put $n-a=a$ in Corollary 3.4, then put λ = 0, we recapture the discrete Montgomery Identity which are cited in Corollary 3.2 of paper [5] and Theorem 2.1 of paper [9].

Continuous Case:

Corollary 3.7. We let $=\mathbb{Z}$. *Then*

$$(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}=\frac{1}{k-j}\int_j^k g(\tau)d\tau+\frac{1}{k-j}\int_j^k p(\theta,\tau)g'(\tau)d\tau\,.$$

Remark 3.8. If we put $j+k-\theta=\theta$ in Corollary 3.7, we recapture the integral Montgomery Identity with parameter which are cited in Corollary 3.5 of [21] and Corollary 3.5 of [20].

Remark 3.9. If we put $j+k-\theta=\theta$ in Corollary 3.7, then put $\lambda=0$, we recapture the integral Montgomery identity which can be found in [22, p. 565] and Theorem 2.1 of paper [9].

Quantum Calculus Case:

Corollary 3.10. *We let* $\mathbb{T}=q^{\mathbb{N}_0}, q>1, j=q^m, k=q^n$ and $\tau=q^c$ *with* $m<n$. *Then*

$$(1-\lambda)\left(\frac{g(\theta)+g(q^m+q^n-\theta)}{2}\right)+\lambda\left(\frac{g(q^m)+g(q^n)}{2}\right)$$
$$=\frac{1}{q^n-q^m}\sum_{c=m}^{n-1}g(q^{c+1})+\frac{1}{q^n-q^m}\sum_{c=m}^{n-1}[g(q^{c+1})-g(q^c)]p(\theta,q^c),$$

where

$$p(\theta,q^c)=\begin{cases}q^c-\left(q^m+\lambda\dfrac{q^n-q^m}{2}\right), & q^m\le q^c<\theta,\\ q^c-\dfrac{q^m+q^n}{2}, & \theta\le q^c<q^m+q^n-\theta,\\ q^c-\left(q^n-\lambda\dfrac{q^n-q^m}{2}\right), & q^m+q^n-\theta\le q^c\le q^n.\end{cases}$$

Remark 3.11. If we put $q^m+q^n-\theta=\theta$ Corollary 3.10, we recapture the Montgomery Identity with parameter in Quantum calculus case which are cited in Corollary 3.7 of [21] and Corollary 3.7 of [20].

Remark 3.12. If we put $q^m+q^n-\theta=\theta$ in Corollary 3.10, then put $\lambda=0$, we recapture the Montgomery identity in Quantum calculus case which can be found in Corollary 3.4 of [5].

Theorem 3.13. *Let* $j,k,\tau,\theta\in\mathbb{T}, j<k$ *and* $g:[j,k]\to\mathbb{R}$ *be differentiable and* $\lambda\in[0,1]$. *If* g^{Δ} *is rd-continuous and*

$$\gamma\le g^{\Delta}(\theta)\le\Gamma \qquad for\ all\ \ \theta\in[j,k].$$

Then we have

$$\left|(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau\right.$$
$$\left.-\frac{g(k)-g(j)}{(k-j)^2}\big(f_2(\theta,j)+f_2(j+k-\theta,\theta)-f_2(j+k-\theta,k)\big)\right|$$
$$\le\frac{1}{4}(k-j)(\Gamma-\gamma) \qquad (3.2)$$

for all $\theta\in[j,k]$.

Proof. By applying Lemma 3.1, we get

$$(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^{\sigma}(\tau)\Delta\tau$$
$$=\frac{1}{k-j}\int_j^k p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau, \qquad (3.3)$$

for all $\theta\in[j,k]$. It is clear that for all $\theta\in[j,k]$ and $\tau\in[j,k]$ we have

$$\theta-k\le p(\theta,\tau)\le\theta-j.$$

Applying Theorem 2.10 to the mapping $p(\theta,\cdot)$ and $g^{\Delta}(\cdot)$, we get

$$\left|\frac{1}{k-j}\int_j^k p(\theta,\tau)g^{\Delta}(\tau)\Delta\tau-\frac{1}{k-j}\int_j^k p(\theta,\tau)\Delta\tau\cdot\frac{1}{k-j}\int_j^k g^{\Delta}(\tau)\Delta\tau\right|$$
$$\le\frac{1}{4}[(\theta-j)-(\theta-k)](\Gamma-\gamma)$$
$$\le\frac{1}{4}(k-j)(\Gamma-\gamma), \qquad (3.2)$$

By a simple calculation we get

$$\int_j^k p(\theta,\tau)\Delta\tau = \int_j^\theta \tau - \left(j+\lambda\frac{k-j}{2}\right)\Delta\tau + \int_\theta^{j+k-\theta}\left(\tau-\frac{j+k}{2}\right)\Delta\tau + \int_{j+k-\theta}^{k}\tau-\left(k-\lambda\frac{k-j}{2}\right)\Delta\tau$$
$$= \int_j^\theta \tau - \left(j+\lambda\frac{k-j}{2}\right)\Delta\tau + \int_\theta^{j+k-\theta}\left(\tau-\frac{j+k}{2}\right)\Delta\tau - \int_k^{j+k-\theta}\tau-\left(k-\lambda\frac{k-j}{2}\right)\Delta\tau$$
$$= f_2(\theta,j) + f_2(j+k-\theta,\theta) - f_2(j+k-\theta,k)$$

and

$$\frac{1}{k-j}\int_j^k g^\Delta(\tau)\Delta\tau = \frac{g(k)-g(j)}{k-j}.$$

By combining (3.3), (3.4) and the above two equalities, we obtain (3.2).

Remark 3.14. If we put $j+k-\theta=\theta$ in Theorem 3.13, we recapture Theorem 3.9 of [21].

Remark 3.15. If we put $j+k-\theta=\theta$ in Theorem 3.13, then put $\lambda = 0$, we recapture Theorem 4 of [15].

Corollary 3.16. *Under the all assumptions of Theorem 3.13 with $\lambda = 1/2$. Then we have*

$$\left|\frac{g(\theta)+g(j+k-\theta)}{4}+\frac{g(j)+g(k)}{4}-\frac{1}{k-j}\int_j^k g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(k-j)(\Gamma-\gamma)$$

for all $\theta \in [j,k]$.

Corollary 3.17. *Under the all assumptions of Theorem 3.13 with $\lambda = 1$. Then we have*

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(k-j)(\Gamma-\gamma),$$

for all $\theta \in [j,k]$.

If g^Δ is bounded on $[j,k]$ then we have the following corollary.

Corollary 3.18. *With the assumptions in Theorem 3.13, if $|g^\Delta(\theta)| \le M$ for all $\theta \in [j,k]$ and some positive constant M, then we have*

$$\left|(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^\sigma(\tau)\Delta\tau - \frac{g(k)-g(j)}{(k-j)^2}\big(f_2(\theta,j)+f_2(j+k-\theta,\theta)-f_2(j+k-\theta,k)\big)\right| \le \frac{1}{4}(k-j)M, \quad (3.5)$$

for all $\theta \in [j,k]$.

Remark 3.19. If we put $j+k-\theta=\theta$ in Corollary 3.18, we recapture Corollary 3.13 of [21].

Remark 3.20. If we put $j+k-\theta=\theta$ in Corollary 3.18, then put $\lambda = 0$, we recapture Corollary 4 of [15].

Furthermore, choosing $\theta = (j+k)/2$ and $\theta = k$, respectively, in (3.2), we have the following corollary.

Corollary 3.21. *With the assumptions in Theorem 3.13, we have*

$$\left|(1-\lambda)g\left(\frac{j+k}{2}\right)+\lambda\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(k-j)(\Gamma-\gamma). \quad (3.6)$$

The above inequality is exactly the same inequality (3.6) on time scale which are cited in Corollary 3.15 of [21] and

$$\left|\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g^\sigma(\tau)\Delta\tau\right| \le \frac{1}{4}(k-j)(\Gamma-\gamma), \quad (3.7)$$

Remark 3.22. If put $\lambda = 0$ in inequality (3.6), we recapture inequality (13) of Corollary 5 of [15].

If we apply the Ostrowski-Grüss like inequality to different time scales, we will get some new results.

Discrete Case:

Corollary 3.23. *We let* $\mathbb{T}=\mathbb{Z}$. *Let* $j=0, k=n, \tau=b, \theta=a$ *and* $g(c)=y_c$. *Then*

$$\left|(1-\lambda)\left(\frac{y_a+y_{n-a}}{2}\right)+\lambda\left(\frac{y_0+y_n}{2}\right)-\frac{1}{n}\sum_{b=1}^{n}y_b-\frac{(y_n-y_0)}{n^2}\sum_{b=0}^{n-1}p(a,b)\right| \leq \frac{1}{4}n(\Gamma-\gamma),$$

where $\gamma \leq \Delta y_a \leq \Gamma$.

Remark 3.24. If we put $n-a=a$ in Corollary 3.23, we recapture the Corollary 3.17 of [21].

Remark 3.25. If we put $n-a=a$ in Corollary 3.23, then put $\lambda=0$, we recapture the Corollary 2 of [15].

Continuous Case:

Corollary 3.26. *We let* $\mathbb{T}=\mathbb{Z}$. *Then we have*

$$\left|(1-\lambda)\left(\frac{g(\theta)+g(j+k-\theta)}{2}\right)+\lambda\frac{g(j)+g(k)}{2}-\frac{1}{k-j}\int_j^k g(\tau)d\tau-\frac{g(k)-g(j)}{(k-j)^2}\int_j^k p(\theta,\tau)d\tau\right| \leq \frac{1}{4}(k-j)(\Gamma-\gamma)$$

for all $\theta \in [j,k]$, *where* $\gamma \leq g'(\theta) \leq \Gamma$.

Remark 3.27. If we put $j+k-\theta=\theta$ in Corollary 3.26, we recapture the Corollary 3.19 of [21].

Remark 3.28. If we put $j+k-\theta=\theta$ in Corollary 3.26, then put $\lambda=0$, we recapture the Corollary 1 of [15].

Quantum Calculus Case:

Corollary 3.29. *We let* $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $j=q^m$ *and* $k=q^n$ *with* $m<n$. *Then*

$$(1-\lambda)\left(\frac{g(\theta)+g(q^m+q^n-\theta)}{2}\right)+\lambda\left(\frac{g(q^m)+g(q^n)}{2}\right)-\frac{1}{q^n-q^m}\sum_{c=m}^{n-1}g(q^{c+1})-\frac{\left(g(q^n)-g(q^m)\right)}{(q^n-q^m)^2}\sum_{c=m}^{n-1}p(\theta,q^c)=\frac{1}{4}(q^n-q^m)(\Gamma-\gamma),$$

where for all $\theta \in [j,k]$.

Remark 3.30. If we put $q^m+q^n-\theta=\theta$ in Corollary 3.29, we recapture the Corollary 3.21 of [21].

Remark 3.31. If we put $q^m+q^n-\theta=\theta$ in Corollary 3.29, then put $\lambda=0$, we recapture the Corollary 3 of [15].

4. **Conclusion.** In this present paper, generalized Ostrowski-Grüss like inequality is proved on time scales using three steps kernel and recapture several established results of different authors of different papers and thus our results unified corresponding to discrete and continuous versions of previously proved results of different researchers in different papers [5, 9, 15, 20, 21, 22]. Moreover, we have used our result to the quantum calculus case.

References

1. Bartosiewicz Z., N. Martins and D. F. M. Torres, The second Euler-Lagrange equation of variational calculus on time scales, Eur. J. Control, 17(1) (2011), 9–18.
2. Bohner M.and A. Peterson, Dynamic equations on time scales, Boston, MA: Birkhäuser Boston Inc., 2001.
3. Bohner M., Calculus of variations on time scales, Dyn. Syst. Appl., 13(3-4) (2004), 339–349.
4. Bohner M. and T. Matthewa, The Grüss inequality on time scales, Communications in Mathematical Analysis, 3(1) (2007), 1–8.
5. Bohner M.and T. Matthewa, Ostrowski inequalities on time scales, J. Inequal. Pure Appl. Math., 9(1) (2008), Art. 6, 8 pp.
6. M. Bohner and S. G. Georgiev, Multivariable Dynamic Calculus on Time Scales, Springer International Publishing Switzerland 2016.

7. Cheng X. L., Improvement of some Ostrowski-Grüss type inequalities, Computers Math. Applic., 42 (2001), 109–114.
8. Dragomir S. S. and S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Comput. Math. Appl., 33 (1997), 15–20.
9. Dragomir S. S., The discrete version of Ostrowski's inequality in normed linear spaces, J. Inequal. Pure Appl. Math., 3(1) (2002), Art. [ONLINE: http://jipam.vu.edu.au/article.php?sid=155].
10. Ferreira R. A. C., A. B. Malinowska and D. F. M. Torres, Optimality conditions for the calculus of variations with higher-order delta derivatives, Appl. Math. Lett., 24(1) (2011), 87–92.
11. Hilger S., Ein MaBkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg (1988).
12. Hilscher R. and V. Zeidan, Calculus of variations on time scales: weak local piecewise C_{rd}^1 solutions with variable endpoints, J. Math. Anal. Appl., 289(1) (2004), 143–166.
13. Hilscher R. and V. Zeidan, Weak maximum principle and accessory problem for control problems on time scales, Nonlinear Anal., 70(9) (2009), 3209–3226.
14. Liu W. J., C. C. Li and Y. M. Hao, Further generalization of some double integral inequal- ities and applications, Acta. Math. Univ. Comenianae, 77(1) (2008), 147–154.
15. Liu W. and Q. A. Ngô, An Ostrowski-Grüss type inequality on time scales, Comput. Math. Appl., 58(2009), 1207—1210.
16. Liu W. J., Q. L. Xue and S. F. Wang, Several new perturbed Ostrowski-like type inequal- ities, J. Inequal. Pure Appl. Math., 8(4) (2007), Art.110, 6 pp.
17. Malinowska A. B., N. Martins and D. F. M. Torres, Transversality conditions for infinite horizon variational problems on time scales, Optim. Lett., 5(1) (2011), 41–53.
18. Malinowska A. B. and D. F. M. Torres, Natural boundary conditions in the calculus of variations, Math. Methods Appl. Sci., 33(2010), 1712–1722.
19. Faraz Mehmood, Asif R. Khan and Muhammad Awais Shaikh, Generalization of Grüss Inequality on Time Scales, Submitted.
20. Faraz Mehmood, Asif R. Khan and Muhammad Awais Shaikh, Generalization of Ostrowski Inequalities on Time Scales, Submitted.
21. Faraz Mehmood, Asif R. Khan and Muhammad Awais Shaikh, Generalization of Ostrowski-Grüss Like Inequality on Time Scales, Submitted.
22. Mitrinović D. S., J. E. Pecˇarić and A. M. Fink, Inequalities involving functions and their integrals and derivatives, vol. 53 of Mathematics and its Applications (East European Series). Dordrecht: Kluwer Academic Publisher Group, p. 565, 1991.
23. Mitrinović D. S., J. Pecarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic, Dordrecht, (1993).
24. Mitrinović D. S., J. Pecarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic, Dordrecht, (1994).
25. Matić M., J. Pecarić and N. Ujević, Improvement and further generalization of inequalities of Ostrowski-Grüss type, Computers Math. Applic., 39(3/4) (2000), 161–175.