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PARTIALLY-ISOSPECTRAL STURM-LIOUVILLE OPERATORS AND THEIR APPLICATIONS

O.Mirzaev[1], R.Abdunazarov[2], G.Nosirova[3]
[1,3]Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.
[2]Jizzakh branch of National University of Uzbekistan named after Mirzo Ulug'bek, Jizzakh, Uzbekistan.

O.Mirzaev Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.
Email: olim-mirzaev@mail.ru

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Abstract. This paper is devoted to the reconstruction of partially-isospectral Sturm-Liouville boundary value problems in a finite segment. Proposed algorithm is applied to find the solution of mixed problems for partial differential equations of parabolic and hyperbolic type with variable coefficients, as well as to find the solution of mixed problems for the fractional-order narrow-band oscillation equation with variable coefficients and the fractional-order single-variable diffusion equations.

Keywords: eigenvalues, integral equation, inverse spectral problem, normalizing constants, partially-isospectral operators, spectral data, Sturm–Liouville problem.

Qisman-izospektral Shturm-Liuvill operatorlari va ularning tatbiqlari

Annotatsiya. Ushbu maqola chekli kesmada qisman-izospektral Shturm-Liuvill chegaraviy muammolarini tiklashga bag'ishlangan. Bu algoritm o'zgaruvchan koeffisiyentli xususiy hosilali parabolik va giperbolik tipli tenglamalarga qo'yilgan aralash masalalarni, shuningdek o'zgaruvchan koeffisiyentli vaqt bo'yicha kasr tartibli tebranish tenglamasi va diffuziya tenglamasiga qo'yilgan aralash masalalarni yechimlarini topishga tatbiq qilingan.

Kalit so'zlar: xos qiymatlar, integral tenglama, teskari spektral masala, normallashtiruvchi o'zgarmaslar, qisman-izospektral operatorlar, spektral ma'lumotlar, Shturm-Liuvill masalasi.

Частично-изоспектральные операторы Штурма-Лиувилля и их приложения

Аннотация. Работа посвящена реконструкции частично-изоспектральных краевых задач Штурма-Лиувилля на конечном отрезке. Предложенный алгоритм применяется для нахождения решения смешанных задач для уравнений в частных производных параболического и гиперболического типа с переменными коэффициентами, а также для нахождения решения смешанных задач для уравнения узкополосных колебаний дробного порядка с переменными коэффициентами и одномерных уравнений диффузии дробного порядка.

Ключевые слова: собственные значения, интегральное уравнение, обратная спектральная задача, нормирующие константы, частично-изоспектральные операторы, спектральные данные, задача Штурма-Лиувилля.

1. Introduction. In this work, we describe all boundary value Sturm-Liouville problems on a finite segment with the same spectrum. Such problems are called isospectral problems and they were studied in works by E.L.Isaacson, H.P.McKean, B.E.Dahlberg, E.Trubowitz ([4]-[7]).

Definition 1. The following Sturm-Liouville boundary value problems:

$$L^0 y \equiv -y'' + q_0(x)y = \lambda y, y(0) = 0, y(\pi) = 0 \quad (1)$$

$$Ly \equiv -y'' + q(x)y = \lambda y, y(0) = 0, y(\pi) = 0 \quad (2)$$

are called isospectral if they have same eigenvalues, that is, $\sigma(L) = \sigma(L^0) = \{\lambda_n^0, n \geq 0\}$.

Definition 2. Boundary value problems (1) and (2) are called partially-isospectral problems if their eigenvalues satisfy the conditions

$$\lambda_n \neq \lambda_n^0 \text{ at } n = \overline{1, n_0}; \ \lambda_n = \lambda_n^0 \text{ at } n = \overline{n_0 + 1, \infty},$$

where $n_0 \in N$ is some natural number.

Here $q_0(x)$, $q(x) \in C[0, \pi]$ are real continuous functions on the segment $[0, \pi]$.

Definition 3. The left-sided and the right-sided Riemann-Liouville fractional integrals $I_{a^+}^{\alpha}$ and $I_{b^-}^{\alpha}$ of order $\alpha \in R^+$ are defined by

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \ t \in (a, b],$$

and

$$I_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} f(s) ds, \ t \in [a, b),$$

respectively. Here $\Gamma(\alpha)$ is a Gamma function.

Definition 4. The left-sided and the right-sided Caputo fractional derivatives ${}^c D_{a^+}^{\alpha}$ and ${}^c D_{b^-}^{\alpha}$ are defined by

$${}^c D_{a^+}^{\alpha} f(t) = I_{a^+}^{n-\alpha} \circ D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ t > a,$$

and

$${}^c D_{b^-}^{\alpha} f(t) = (-1)^n I_{b^-}^{n-\alpha} \circ D^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha+1} f^{(n)}(s) ds, \ t < b,$$

respectively, where f is a sufficiently differentiable function and $n-1 < \alpha < n$.

Definition 5. The left-sided and the right-sided Riemann-Liouville fractional derivatives $D_{a^+}^{\alpha}$ and $D_{b^-}^{\alpha}$ are defined by

$$D_{a^+}^{\alpha} f(t) = D^n \circ I_{a^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ t > a,$$

and

$$D_{b^-}^{\alpha} f(t) = (-1)^n D^n \circ I_{b^-}^{n-\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s) ds, \ t < b,$$

respectively, where f is a sufficiently differentiable function and $n-1 < \alpha < n$.

Definition 6. The function $E_{\alpha}(z)$ defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \ (z \in C, \Re(\alpha) > 0),$$

was introduced by Mittag-Leffler [26-27]. In particular, when $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(z) = e^z, \ E_2(z) = \cosh(\sqrt{z}).$$

The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}$$

where $z, \beta \in C$ and $\mathrm{Re}(\alpha) > 0$. It coincides with the Mittag-Leffler function for $\beta = 1$:

$$E_{\alpha,1}(z) = E_{\alpha}(z).$$

In particular, when $\alpha = 1, \beta = 2$ and $\alpha = 2, \beta = 2$, we have

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \; E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.$$

Nowadays, there are various methods for solving inverse spectral problems, such as, the method of transformation operator, that is, Gelfand-Levitan method, the method of spectral mappings, the method of etalon models and others ([8]-[22]). Marchenko proved that the Sturm-Liouville operator on a finite segment determined uniquely by its eigenvalues and a sequence of normalizing constants, that is, by its spectral functions. Gelfand and Levitan found necessary and sufficient conditions on recovering boundary value Sturm-Liouville problems by their spectral functions. This method is based on recovering a potential and boundary conditions by spectral data by means of a second kind Fredholm integral equation with parameters. While recontructing partially-isospectral boundary value Sturm-Liouville problems with a discret spectrum λ_n, $n \geq 0$, we have employed the Gelfand-Levitan method.

The main result of the work is an algorithm for recovering a family of boundary value Sturm-Liouville problems $L = L(q(x), a, b), \infty, \infty)$, whose spectrum satisfy the conditions

$$\sigma(L) = \left\{\lambda_1 = a^2, a \in (-1,1), \lambda_n = n^2, n \geq 2\right\} \text{ and } \sigma(L) = \left\{\lambda_1 = -a^2, a \in R, \lambda_n = n^2, n \geq 2\right\}.$$

This algorithm has also been applied to solving mixed problems for partial differential equations of parabolic and hyperbolic type with variable coefficients, as well as to solving mixed problems for fractional-order narrow-band oscillation equations with variable coefficients and fractional-order single-variable diffusion equations.

2. Fundamental notations regarding the inverse spectral problem. We consider the following boundary value problem:

$$L(q(x), \infty, \infty)y \equiv -y'' + q(x)y = \lambda y, (0 < x < \pi), \tag{3}$$

$$y(0) = 0, \tag{4}$$

$$y(\pi) = 0, \tag{5}$$

where $q(x) \in C[0, \pi]$, λ is a spectral parameter.

Let $\varphi(x, \lambda)$ represents a solution to equation (3), which satisfies the initial conditions

$$\varphi(0, \lambda) = 0, \varphi'(0, \lambda) = 1. \tag{6}$$

As established in [3], there exists a unique solution, denoted as $\phi(x, \lambda)$, to the problem (3), (6). For any fixed $x \in [0, \pi]$, this solution is an entire function in λ. Furthermore, the following integral representation holds:

$$\varphi(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x K(x,t)\sin\sqrt{\lambda}t dt, \tag{7}$$

where

$$K(x, x) = \frac{1}{2}\int_0^x q(t)dt. \tag{8}$$

It is known that, for any λ, the function $\phi(x, \lambda)$ satisfies the boundary condition (4). Consequently, the eigenvalues λ_n, $n = 1, 2, 3, \ldots$, in the problem defined by equations (3)-(5) correspond to the roots of the equation

$$\Delta(\lambda) \equiv \varphi(\pi, \lambda) = 0, \tag{9}$$

where $\varphi(x, \lambda_n)$ represents the sequence of eigenfunction for $n = 1, 2, 3, \ldots$.

We define

$$\alpha_n = \int_0^{\pi} \varphi^2(x, \lambda_n)dx. \tag{10}$$

The constants α_n is a normalizing constants of the boundary value problem (3)-(5). Subsequently, the set of numbers $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ is called the spectral data of the problem (3)-(5).

Theorem 1 ([3],[9]). The spectral data $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ of problem (3)-(5) saisfies the identities

$$\sqrt{\lambda_n} = n + \frac{c}{\pi n} + \frac{\gamma_n}{n^2},\ \{\gamma_n\} \in l_2,\quad \alpha_n = \frac{\pi}{2n^2} + \frac{\beta_n}{n^3},\ \{\beta_n\} \in l_2, \tag{11}$$

$$c = \frac{1}{2}\int_0^{\pi} q(t)dt. \tag{12}$$

The orthogonality of eigenfunctions corresponding to distinct eigenvalues is a well-known fact, and for any function $f(x) \in L_2(0,\pi)$ it can be expressed as the series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \left(\int_0^{\pi} f(t)\varphi(t,\lambda_n)dt\right)\varphi(x,\lambda_n). \tag{13}$$

This indicates a symbolic sense of identity:

$$\sum_{n=1}^{\infty} \frac{\varphi(t,\lambda_n)\varphi(x,\lambda_n)}{\alpha_n} = \delta(t-x), \tag{14}$$

where $\delta(x)$ is a Dirac delta function. Specifically, when $q(x) \equiv 0$ we have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^0} \cdot \frac{\sin nx \sin nt}{n^2} = \delta(t-x), \tag{15}$$

where

$$\alpha_n^0 = \frac{\pi}{2n^2},\ n \geq 1. \tag{16}$$

Theorem 2 (V.A.Marchenko [1]). The potential $q(x)$ of the boundary value problem (3)-(5) is uniquely determined by the spectral data $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$.

Lemma 1 ([2]). The following identity holds:

$$\sum_{n=1}^{\infty} \frac{\varphi(x,\lambda_n)}{\alpha_n} \cdot \frac{\sin\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} = 0,\ 0 < t < x. \tag{17}$$

Theorem 3 ([2]). The kernel $K(x,t)$ of the transformation operator (2.5) satisfies the integral equation

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0,\ (0 < t < x), \tag{18}$$

where

$$F(x,t) = \sum_{n=1}^{\infty} \left\{\frac{1}{\alpha_n} \cdot \frac{\sin\sqrt{\lambda_n}x \sin\sqrt{\lambda_n}t}{\lambda_n} - \frac{1}{\alpha_n^0} \cdot \frac{\sin nx \sin nt}{n^2}\right\}. \tag{19}$$

Theorem 4 ([2],[9]). A set of real eigenvalues $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ be a spectral data for a boundary value problem (3)-(5) with a potential $q(x) \in L^2(0,\pi)$ if and only if conditions (11) and (12) are satisfied.

Suppose a sequence $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ satisfy the condition (11). We find a function $F(x,t)$ using formula (19) and examine the family of integral equations (18) for $K(x,t)$.

Theorem 5 ([2]). For each fixed $x \in (0,\pi)$, integral equation (18) has a unique solution $K(x,t) \equiv K_x(t)$.

By solving equation (18), we determine $K(x,t)$. Subsequently, we define a function $\varphi(x,\lambda)$ using formula (7). The function $\varphi(x,\lambda)$ satisfies the differential equations

$$-\varphi'' + q(x)\varphi = \lambda\varphi,\ (0 < x < \pi), \tag{20}$$

with the following initial conditions:

$$\varphi(0,\lambda)=0,\ \varphi'(0,\lambda)=1, \tag{21}$$

where

$$q(x)=2\frac{d}{dx}K(x,x). \tag{22}$$

3. Algorithm of recovering partially-isospectral boundary value problems.

Theorem 6. If the sequence of real numbers $\{\lambda_n,\alpha_n\}_{n=1}^{\infty}$ satisfies the following conditions:

$$\lambda_1=a^2, a\in(-1;1), \lambda_n=n^2, \alpha_1=b>0, \alpha_n=\frac{\pi}{2n^2}, n\geq 2 \tag{23}$$

then this pair is the spectral data of some Sturm-Liouville boundary value problem of the form $L(q(x,a,b),\infty,\infty)$.

Proof. It is easy to check that the sequence $\{\lambda_n,\alpha_n\}_{n=1}^{\infty}$ defined by identities (23), satisfies the conditions (11) and (12), that is, $c=0,\gamma_n=0,\beta_n=0$. Therefore, there exists a unique boundary value problem of form $L(q(x,a,b),\infty,\infty)=L(a,b)$ with the coefficients $q(x,a,b)\in L_2(0,\pi)$.

To find the coefficients $q(x,a,b)$ of boundary value problem $L(q(x,a,b),\infty,\infty)=L(a,b)$, we use the Gelfand-Levitan method for solving the inverse spectral problem.

For this, first, we define $F(x,t)$ by formulas (19) and (23):

$$F(x,t)=\frac{\sin ax\sin at}{a^2b}-\frac{2\sin x\sin t}{\pi}. \tag{24}$$

Then we substitute (24) into integral equation (18) to obtain

$$K(x,t)=-F(x,t)-\int_0^x K(x,s)F(s,t)ds=-\frac{\sin at}{a^2b}f(x)+\frac{2\sin t}{\pi}g(x), \tag{25}$$

where

$$f(x)=\sin ax+\int_0^x K(x,s)\sin as ds, \tag{26}$$

$$g(x)=\sin x+\int_0^x K(x,s)\sin s ds. \tag{27}$$

Substituting expression (25) into formulas (26) and (27), we obtain

$$f(x)=\sin ax-\frac{f(x)}{a^2b}\int_0^x \sin^2 as ds+\frac{2g(x)}{\pi}\int_0^x \sin as\sin s ds, \tag{28}$$

$$g(x)=\sin x-\frac{f(x)}{a^2b}\int_0^x \sin as\sin s ds+\frac{2g(x)}{\pi}\int_0^x \sin^2 s ds. \tag{29}$$

Thus

$$f(x)=\frac{4a^3b[(1-a^2)^2\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax\sin 2ax)(2\pi+2x-\sin 2x)-32a(a\sin x\cos ax-\sin ax\cos x)^2}, \tag{30}$$

$$g(x)=\frac{2\pi(1-a^2)[(1-a^2)\sin x(4a^3b+2ax\sin 2ax)-8a\sin ax(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax\sin 2ax)(2\pi+2x-\sin 2x)-32a(a\sin x\cos ax-\sin ax\cos x)^2}. \tag{31}$$

Then it follows from (25), (30) and (31) that

$$K(x,t)=-\frac{4a\sin at[(1-a^2)^2\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2}+$$
$$+\frac{2(1-a^2)\sin t[(1-a^2)\sin x(4a^3b+2ax\sin 2ax)-8a\sin ax(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2}, \tag{32}$$

In addition, taking into account formulas (22) and (32), we have

$$K(x,x)=-\frac{4a\sin ax[(1-a^2)^2\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax\sin 2ax)(2\pi+2x-\sin 2x)-32a(a\sin x\cos ax-\sin ax\cos x)^2}$$
$$+\frac{2(1-a^2)\sin x[(1-a^2)\sin x(4a^3b+2ax\sin 2ax)-8a\sin ax(a\sin x\cos ax-\sin ax\cos x)]}{(1-a^2)(4a^3b+2ax\sin 2ax)(2\pi+2x-\sin 2x)-32a(a\sin x\cos ax-\sin ax\cos x)^2},$$
$$\frac{dK(x,x)}{dx}=-\frac{4a[a(1-a^2)\sin 2ax(2\pi+2x-\sin 2x)-4(a^2\sin^2 x\cos^2 ax-\sin^2 ax\cos^2 x)]}{(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2}$$
$$+\frac{4(1-a^2)[(1-a^2)\sin 2x(4a^3b+2ax-\sin 2ax)-8a(a^2\sin^2 x\cos^2 ax-\sin^2 ax\cos^2 x)]}{(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2}$$
$$+\frac{16a(1-a^2)\sin ax[(1-a^2)\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)]}{[(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2]^2}$$
$$\times[a(2\pi+2x-\sin 2x)\sin^2 ax+(4a^3b+2ax-\sin 2ax)\sin^2 x-4a(a\sin^2 x\sin 2ax-\sin^2 ax\sin 2x)]$$
$$-\frac{16(1-a^2)^2\sin x[(1-a^2)\sin x(4a^3b+2ax-\sin 2ax)-4a\sin ax(a\sin x\cos ax-\sin ax\cos x)]}{[(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2]^2}$$
$$\times[a(2\pi+2x-\sin 2x)\sin^2 ax+(4a^3b+2ax-\sin 2ax)\sin^2 x-4a(a\sin^2 x\sin 2ax-\sin^2 ax\sin 2x)],$$
$$q(x)=q(x,a,b)=2K'(x,x). \qquad (33)$$

Thus, it was possible to find all the coefficients of the family of isospectral Sturm-Liouville boundary value problems of the following form:

$$L(a,b)y\equiv -y''+q(x,a,b)y=\lambda y,\ 0<x<\pi,$$
$$y(0)=0,\ y(\pi)=0. \qquad (34)$$

All eigenvalues of the boundary value problems (34) consist of given numbers $\lambda_1=a^2$, $a\in(-1;1)$, $\lambda_n=n^2$, $n\geq 2$, and the corresponding eigenfunctions are found by the formulas

$$y_1(x,\lambda)=\frac{\sin ax}{a}$$
$$-\frac{2[(1-a^2)^2\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)][2ax-\sin 2ax]}{a[(1-a^2)(4a^3b+2ax\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2}$$
$$+\frac{[(1-a^2)\sin x(4a^3b+2ax\sin 2ax)-8a\sin ax(a\sin x\cos ax-\sin ax\cos x)][a\sin x\cos ax-\sin ax\cos x]}{a[(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-32a(a\sin x\cos ax-\sin ax\cos x)^2]},$$
$$y_n(x,\lambda)=\frac{\sin nx}{n}$$
$$-\frac{2a[(1-a^2)^2\sin ax(2\pi+2x-\sin 2x)-4\sin x(a\sin x\cos ax-\sin ax\cos x)]}{n[(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2]}$$
$$\times[\frac{\sin(n-a)t}{n-a}-\frac{\sin(n+a)t}{n+a}]$$
$$+\frac{(1-a^2)[(1-a^2)\sin x(4a^3b+2ax\sin 2ax)-8a\sin ax(a\sin x\cos ax-\sin ax\cos x)]}{n[(1-a^2)(4a^3b+2ax-\sin 2ax)(2\pi+2x-\sin 2x)-16a(a\sin x\cos ax-\sin ax\cos x)^2]}$$
$$\times[\frac{\sin(n-1)t}{n-1}-\frac{\sin(n+1)t}{n+1}].$$

Now, using isospectral boundary value problems (34), we find a family of oscillatory systems (strings) with a given frequency:

$$\square(L_{a,b})u\equiv\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+q(x,a,b)\right)u=0,\ 0\leq a<1,b>0, \qquad (35)$$

and the following diffusion equations:

$$\Im(L_{a,b})u \equiv \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + q(x,a,b) \right) u = 0,\ 0 \le a < 1, b > 0 \quad . \tag{36}$$

Here $q(x,a,b)$ is defined by the formula (33).

Using Theorem 6, we obtain the following results.

Corollary 1. Solution of the mixed problem for a hyperbolic type equation with variable coefficients (35):

$$u(x,0) = f_1(x),\ u_t(x,0) = f_2(x),\ f_1(x) \in C^3[0,\pi],\ f_2(x) \in C^2[0,\pi],$$

$$u(0,t) = 0,\quad u(\pi,t) = 0,$$

is as follows:

$$u(x,t) = \left(p_1 \cos at + q_1 \sin at \right) y_1(x) + \sum_{n=2}^{\infty} \left(p_n \cos nt + q_n \sin nt \right) y_n(x) \quad .$$

Here

$$p_1 = \frac{2}{\pi} \int_0^{\pi} f_1(x) y_1(x) dx \quad q_1 = \frac{2}{a\pi} \int_0^{\pi} f_2(x) y_1(x) dx \quad ,$$

$$p_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) y_n(x) dx \quad q_n = \frac{4}{(2n+1)\pi} \int_0^{\pi} f_2(x) y_n(x) dx, n \ge 2 \quad .$$

Corollary 2 . Solving a mixed problem for a parabolic type equation with variable coefficients (36):

$$u(x,0) = f(x),\ f(x) \in C^1[0,\pi],\ u_x'(0,t) = 0,\ u(0,t) = 0,\ u(\pi,t) = 0$$

is as follows:

$$u(x,t) = B_1 e^{-a^2 t} y_1(x) + \sum_{n=2}^{\infty} B_n e^{-n^2 t} y_n(x) \quad .$$

Here $B_1 = \frac{2}{\pi} \int_0^{\pi} f(x) y_1(x) dx \quad B_n = \frac{2}{\pi} \int_0^{\pi} f(x) y_n(x) dx, n \ge 2 \quad .$

Fractional order differential equations and fractional order partial differential equations were studied in [23-24], and the following results were obtained using the above algorithm.

Corollary 3. The solution to the following mixed problem for nonlocal narrow-band oscillation equation with variable coefficients

$$D_{0t}^{\alpha} u(x,t) = u_{xx} - q(x,a,b)u, 1 < \alpha < 2, 1 < \alpha < 2, b > 0, 0 < x < \pi, t > 0 \quad ,$$

$$\lim_{t \to 0} t^{\alpha-2} u(x,0) = f_1(x),\ \lim_{t \to 0} [t^{\alpha-2} u(x,0)]_t = f_2(x),\ f_1(x) \in C^3[0,\pi],\ f_2(x) \in C^2[0,\pi] \quad ,$$

$u(0,t) = 0$, $u(\pi,t) = 0,$ is determined by the formula

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(t) y_n(x)$$

Here $\varphi_n(t) = a_n t^{\alpha-1} E_{1/\alpha}(-\lambda_n t^{\alpha}; \alpha) \Gamma(\alpha) + b_n t^{\alpha-2} E_{1/\alpha}(-\lambda_n t^{\alpha}; \alpha - 1) \Gamma(\alpha - 1)$

$a_n = \lim_{t \to 0} t^{\alpha-2} \varphi_n(t),\ b_n = \lim_{t \to 0} [t^{\alpha-2} \varphi_n(t)]_t$, $E_{1/\alpha}(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \alpha k)}$ is a Mittag-Leffler function. $\Gamma(z) = z!$ is a Gamma function.

Corollary 4. The solution to the following mixed problem for nonlocal narrow-band oscillation equation with variable coefficients

$$D_{0t}^{\alpha} u(x,t) = u_{xx} - q(x,a,b)u, 0 < \alpha < 1, b > 0, 0 < x < \pi, t > 0$$

$u(x,0) = f_1(x)$, $u(0,t) = 0$, $u(\pi,t) = 0$ is determined by the formula

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(t) y_n(x).$$

Here $\varphi_n(t) = a_n E_{1/\alpha}(-\lambda_n t^{\alpha};\alpha),\ a_n = \frac{2}{\pi}\int_0^{\pi} f_1(x) y_n(x)dx$.

Theorem 7. If the sequence of real numbers $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ satisfies the following conditions:

$$\lambda_1 = -a^2, a \in R,\ \lambda_n = n^2, n \geq 2; \alpha_1 = b, b > 0, \alpha_n = \frac{\pi}{2n^2}, n \geq 2. \tag{37}$$

Then this pair is the spectral data of some Sturm-Liouville boundary value problem of the form $L(q(x,a,b),\infty,\infty)$.

Proof. It is easy to check that the sequence $\{\lambda_n, \alpha_n\}_{n=1}^{\infty}$ defined by identities (37), satisfies the conditions (11) and (12), that is, $c = 0, \gamma_n = 0, \beta_n = 0$. Therefore, there exists a unique boundary value problem of form $L(q(x,a,b),\infty,\infty) = L(a,b)$ with the coefficients $q(x,a,b) \in L_2(0,\pi)$.

To find the coefficients $q(x,a,b)$ of this boundary value problem $L(q(x,a,b),\infty,\infty) = L(a,b)$, we use the Gelfand-Levitan method for solving the inverse spectral problem.

For this, we define $F(x,t)$ by formulas (19) and (37):

$$F(x,t) = \frac{\sinh ax \sinh at}{a^2 b} - \frac{2}{\pi}\sin x \sin t. \tag{38}$$

Then we substitute (38) into integral equation (18) to obtain

$$K(x,t) = -\frac{\sinh at}{a^2 b} f(x) + \frac{2\sin t}{\pi} g(x), \tag{39}$$

where

$$f(x) = \sinh ax + \int_0^x K(x,s)\sinh as ds, \tag{40}$$

$$g(x) = \sin x + \int_0^x K(x,s)\sin x ds. \tag{41}$$

Substituting expression (39) into formulas (40) and (41), we have

$$f(x) = \sinh ax - \frac{f(x)}{a^2 b}\int_0^x \sinh^2 as ds + \frac{2g(x)}{\pi}\int_0^x \sinh as \sin s ds, \tag{42}$$

$$g(x) = \sin x - \frac{f(x)}{a^2 b}\int_0^x \sinh as \sin s ds + \frac{2g(x)}{\pi}\int_0^x \sin^2 s ds. \tag{43}$$

Thus

$$f(x) = \frac{4a^3 b(a^2+1)[\sinh ax(2\pi - 2x + \sin 2x) + 4\sin x(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2}, \tag{44}$$

$$g(x) = \frac{2\pi(a^2+1)[\sin x(4a^3 b - 2ax + \sinh 2ax) - 4a\sinh ax(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2}. \tag{45}$$

Then it follows from (39), (44) and (45) that

$$K(x,t) = -\frac{4a(a^2+1)\sinh at[\sinh ax(2\pi - 2x + \sin 2x) + 4\sin x(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2}$$

$$+\frac{4\pi(a^2+1)\sin t[\sin x(4a^3 b - 2ax + \sinh 2ax) - 4a\sinh ax(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2}. \tag{46}$$

Furthermore, taking into account formulas (22) and (46), we have

$$K(x,x) = -\frac{4a(a^2+1)\sinh ax[\sinh ax(2\pi - 2x + \sin 2x) + 4\sin x(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2} +$$

$$+\frac{4\pi(a^2+1)\sin x[\sin x(4a^3 b - 2ax + \sinh 2ax) - 4a\sinh ax(a\cosh ax \sin x - \sinh ax \cos x)]}{(a^2+1)^2(2\pi - 2x + \sin 2x)(4a^3 b - 2ax + \sinh 2ax) + 16a(a\cosh ax \sin x - \sinh ax \cos x)^2},$$

$$\frac{dK(x,x)}{dx}=\frac{4(a^2+1)}{(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2}$$
$$\times\{-a^2\sinh 2ax(2\pi-2x+\sin 2x)-4a(1+\pi)(a^2\cosh^2 ax\sin^2 x-\sinh^2 ax\cos^2 x)$$
$$-4a^3(1+\pi)\sin^2 x\sinh^2 ax+\pi\sin 2x(4a^3b-2ax+\sinh 2ax)\}$$
$$+\frac{4(a^2+1)^2}{[(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2]^2}$$
$$\times\{(a^2+1)[4a\sinh^2 ax(2\pi-2x+\sin 2x)-4\sin^2 x(4a^3b-2ax+\sinh 2ax)]$$
$$+16a(a\sinh 2ax\sin^2 x-\sinh^2 ax\sin 2x)\}$$
$$\times[a\sinh^2 ax(2\pi-2x+\sin 2x)+2a(1+\pi)(a\sinh 2ax\sin^2 x-\sinh^2 ax\sin 2x)$$
$$-\pi\sin^2 x(4a^3b-2ax+\sinh 2ax)],$$
$$q(x)\equiv q(x,a)=2K'(x,x). \qquad (47)$$

Thus, it was possible to find all the coefficients of the family of isospectral Sturm-Liouville boundary value problems of the following form:

$$L(a,b)y\equiv -y''+q(x,a,b)y=\lambda y,\ 0<x<\pi,$$
$$y(0)=0,\ y(\pi)=0. \qquad (48)$$

All eigenvalues of the boundary value problem (48) consist of given numbers $\lambda_1=-a^2$, $a\in R$, $\lambda_n=n^2$, $n\geq 2$, and the corresponding eigenfunctions are found by the formulas

$$y_1(x)=\frac{\sinh ax}{a}$$
$$-\frac{(a^2+1)[\sinh ax(2\pi-2x+\sin 2x)+2(2a\cosh ax\sin^2 x-\sinh ax\sin 2x)][2ax-\sin 2ax]}{(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2}$$
$$+\frac{4\pi[\sin x(4a^3b-2ax+\sinh 2ax)-2a(a\sinh 2ax\sin x-2\sinh^2 ax\cos x)][a\cosh ax\sin x-\sinh ax\cos x]}{a[(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2]},$$
$$y_n(x)=\frac{\sin nx}{n}$$
$$-\frac{4a(a^2+1)[\sinh ax(2\pi-2x+\sin 2x)+2(2a\cosh ax\sin^2 x-\sinh ax\sin 2x)][a\cosh ax\sin x-n\sinh ax\cos x]}{n(a^2+n^2)[(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2]}$$
$$+\frac{4\pi(a^2+1)[\sin x(4a^3b-2ax+\sinh 2ax)-2a(a\sinh 2ax\sin x-2\sinh^2 ax\cos x)][n\sin x\cos nx+\sin nx\cos x]}{n(n^2-1)[(a^2+1)^2(2\pi-2x+\sin 2x)(4a^3b-2ax+\sinh 2ax)+16a(a\cosh ax\sin x-\sinh ax\cos x)^2]},$$

Now, using isospectral boundary value problems (48), we find a family of oscillatory systems (strings) with a given frequency:

$$\square(L_{a,b})u\equiv\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+q(x,a,b)\right)u=0,\ a\in R,b>0 \qquad (49)$$

and the following diffusion equations:

$$\Im(L_{a,b})u\equiv\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+q(x,a,b)\right)u=0,\ a\in R,b>0. \qquad (50)$$

Here $q(x,a,b)$ is determined by the formula (48).

Using Theorem 7, we obtain the following results.

Corollary 5. Solving a mixed problem for a hyperbolic type equation with variable coefficients (49):

$$u(x,0)=f_1(x),\ u_t(x,0)=f_2(x),\ f_1(x)\in C^3[0,\pi],\ f_2(x)\in C^2[0,\pi],$$
$$u(0,t)=0,\ u(\pi,t)=0$$ is as follows:

$$u(x,t)=\left(p_1\cosh at+q_1\sinh at\right)y_1(x)+\sum_{n=2}^{\infty}\left(p_n\cos nt+q_n\sin nt\right)y_n(x)$$

.

Here

$$p_1=\frac{2}{\pi}\int_0^{\pi}f_1(x)y_1(x)dx \quad q_1=\frac{2}{a\pi}\int_0^{\pi}f_2(x)y_1(x)dx$$

,

$$p_n=\frac{2}{\pi}\int_0^{\pi}f_1(x)y_n(x)dx \quad q_n=\frac{4}{(2n+1)\pi}\int_0^{\pi}f_2(x)y_n(x)dx, n\geq 2$$

,

.

Corollary 6 . Solving a mixed problem for a parabolic type equation with variable coefficients (50):

$u(x,0)=f(x),\ f(x)\in C^1[0,\pi]$, $u(0,t)=0$, $u(\pi,t)=0$ is as follows:

$$u(x,t)=B_1e^{a^2t}y_1(x)+\sum_{n=2}^{\infty}B_ne^{-n^2t}y_n(x)$$

.

Here

$$B_1=\frac{2}{\pi}\int_0^{\pi}f(x)y_1(x)dx \quad B_n=\frac{2}{\pi}\int_0^{\pi}f(x)y_n(x)dx, n\geq 2$$

,

.

Corollary 7. The solution to the following mixed problem for nonlocal narrow-band oscillation equation with variable coefficients

$$D_{0t}^{\alpha}u(x,t)=u_{xx}-q(x,a,b)u, 1<\alpha<2, 1<\alpha<2, b>0, 0<x<\pi, t>0,$$

$$\lim_{t\to 0}t^{\alpha-2}u(x,0)=f_1(x),\ \lim_{t\to 0}[t^{\alpha-2}u(x,0)]_t=f_2(x),\ f_1(x)\in C^3[0,\pi],\ f_2(x)\in C^2[0,\pi]$$

,

$u(0,t)=0$, $u(\pi,t)=0,$ is determined by the formula

$$u(x,t)=\sum_{n=1}^{\infty}\varphi_n(t)y_n(x)$$

.

Corollary 8. The solution to the following mixed problem for nonlocal narrow-band oscillation equation with variable coefficients

$$D_{0t}^{\alpha}u(x,t)=u_{xx}-q(x,a,b)u, 0<\alpha<1, b>0, 0<x<\pi, t>0$$

$$u(x,0)=f_1(x),\ u(0,t)=0,\ u(\pi,t)=0$$

is determined by the formula

$$u(x,t)=\sum_{n=1}^{\infty}\varphi_n(t)y_n(x).$$

Here

$$\varphi_n(t)=a_nE_{1/\alpha}(-\lambda_nt^{\alpha};\alpha),\ a_n=\frac{2}{\pi}\int_0^{\pi}f_1(x)y_n(x)dx$$

.

Conclusion. In conclusion, using the inverse problem method, one can reconstruct families of partially-isospectral Sturm-Liouville boundary value problems whose eigenvalues coincide with the given numbers:

$\sigma(L(a,b))=\left\{\lambda_1=a^2, a\in[0;1), \lambda_n=n^2, n\geq 2\right\}$ and $\sigma(L(a,b))=\left\{\lambda_1=-a^2, a\in R, \lambda_n=n^2, n\geq 2\right\}$.

In addition, using the proposed algorithm, we find the solution of mixed problems for partial differential equations of parabolic and hyperbolic type with variable coefficients, as well as to find the solution of mixed problems for the fractional-order narrow-band oscillation equation with variable coefficients and the fractional-order single-variable diffusion equations.

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