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NEW EQUIVALENT CONDITIONS FOR DISCRETE HARDY-TYPE INEQUALITIES

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Abstract. In this work, a new pair of necessary and sufficient conditions for the validity of discrete Hardy-type inequality with Oinarov's kernel is given. In addition, lower and upper estimates are obtained for the best constant (the smallest among the constants for which the inequality is valid) of the inequality.

Keywords: equivalent conditions, discrete inequality, Hardy inequality, kernel, weight.

Diskret Hardy tipidagi tengsizliklar uchun yangi ekvivalent shartlar

Annotatsiya. Ushbu ishda Oynarov yadroli diskret Hardy tipidagi tengsizlik o‘rinli bo‘lishi uchun zarur va yetarli shartlar juftligi keltirilgan. Bundan tashqari tengsizlikning eng yaxshi konstantasi (tengsizlik o‘rinli bo‘ladigan konstantalar ichida eng kichigi) uchun quyi va yuqori baholar olingan.

Kalit so‘zlar: ekvivalent shartlar, diskret tengsizliklar, Hardy tengsizligi, yadro, vazn.

Новые эквивалентные условия для дискретных неравенств типа Харди

Аннотация. В данной работе дана новая пара необходимых и достаточных условий справедливости дискретного неравенства типа Харди с ядром Ойнарова. Кроме того, получены нижняя и верхняя оценки для лучшей константы неравенства (наименьшей среди констант, для которых справедливо неравенство).

Ключевые слова: эквивалентные условия, дискретные неравенства, неравенства Харди, ядро, вес.

Introduction. Let us consider the following discrete weighted Hardy inequality with kernel for nonnegative sequence $f = \{f_n\}_{n=1}^{\infty}$ in the form

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} a_{n,k} f_k\right)^q\right)^{\frac{1}{q}} \le C \left(\sum_{n=1}^{\infty} v_n^p f_n^p\right)^{\frac{1}{p}}, \qquad (1)$$

where $1 < p, q < \infty$, $u = \{u_n\}_{n=1}^{\infty}$ and $v = \{v_n\}_{n=1}^{\infty}$ positive sequences of real numbers and

$$(Af)_n = \sum_{k=1}^{n} a_{n,k} f_k$$

is a matrix operator with the kernel $a := \{a_{n,k}\}_{n,k=1}^{\infty}$, $n \ge k$, such that $a_{n,k} \ge 0$ for $n \ge k \ge 1$. Let $l_{p,v}$ denote the space of all sequence $f = \{f_n\}_{n=1}^{\infty}$ of real numbers whose norm $\|f\|_{l_{p,v}} = (\sum_{n=1}^{\infty} |v_n f_n|^p)^{\frac{1}{p}}$ is finite.

If $a_{n,k} \equiv 1$, then (1) takes the form

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} f_k\right)^q\right)^{\frac{1}{q}} \le C\left(\sum_{n=1}^{\infty} v_n^p f_n^p\right)^{\frac{1}{p}} \qquad (2)$$

is called *generalized Hardy inequality*. In this case the problem of finding necessary and sufficient conditions for the inequality (2) to be hold is completely solved in the parameters $1 < p < \infty$ and $0 < q < \infty$, in addition, lower and upper estimates of various forms for the best constant C of the inequality (2) were also obtained, (see [1-2]).

If the kernel of the inequality (1) is different from a constant sequence, then the investigation of the inequality becomes a bit more difficult. In the last decades of the last century, many scientists: K.F.Andersen, H.P.Heinig, M.Sh.Braverman, V.D.Stepanov, R.Oinarov, G.Sinnamon, L-E.Persson and others studied the inequality (1) and also its *integral form* with kernels satisfying some additional conditions in the parameters $1 < p < \infty$ and $0 < q < \infty$ (see [2-6]). The main goal of the scholars mentioned above was to find equivalent conditions for the inequality to be hold. Therefore, the results of finding the best constant of (1) or its lower and upper estimates are very rare. Such results began to appear recently for continuous, i.e., integral inequalities. In the obtained results, it can be seen that finding *new equivalent conditions* and using them to estimate the constant gave better results.

Initially, new equivalent conditions for the inequality (2) were obtained by G.Bennet [7] in the case $1 < p \le q < \infty$. Later, L.-E.Persson and V.Stepanov [8] also obtained similar conditions for the *integral form* of (2) in the case $1 < p < \infty$ and $0 < q < \infty$. Conditions related to a parameter were presented by A.Wedestig in her PhD thesis [9]. A.Kufner, L-E. Persson, A. Wedestig, A. Gogatishvili, P. Wall, Ch.Okpoti, K.Kuliev and others found various new equivalent conditions (see [10-11]). The foregoing results related to the case when the kernel is a constant. If the kernel is not a constant sequences then new equivalent conditions are very rare. S. Kayser, L. Nikolova, L.-E.Persson and A. Wedestig gave parameter dependent sufficient condition for the integral form of (1) with rather general kernel and upper estimate for its best constant (see [12]). A.Kufner, K.Kuliyev, M.Eshimova and G.Kuliyeva studied the case $1 < p, q < \infty$ in which the authors gave estimates for the best constant with new equivalent conditions [13-16].

In this paper we present a new pair of equivalent condition for satisfying of the inequality (1) with so-called Oinarov's kernel, i.e., for the kernels satisfying: $a_{n,k}$ is increasing by n, decreasing by k and there exists a number $h \ge 1$ such that,

$$a_{n,k} \le h\left(a_{n,s} + a_{s,k}\right), \text{ all } n \ge s \ge k \ge 1.$$

Kernel $\{a_{n,k}\}$ satisfying these conditions is called *Oinarov's kernel.*

Recently, in 2024 A.Kalybay and S. Shalginbayeva [17] obtained new estimates for the best consatnt of the inequality (1). Let us present their result:

Theorem 1.1. (Theorem 2.1. [17]). *Let* $1 < p \le q < \infty$ *and* $\{a_{n,k}\}$ *be Oinarov's kernel. Then for any non-negative* $f \in l_{p,v}$ *inequality (1) holds if and only if* $A = \max\{A_1, A_2\} < \infty$, *where*

$$A_1 = \sup_{k \ge 1}\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{k} v_j^{-p'}\right)^{\frac{1}{p}},$$

$$A_2 = \sup_{k \ge 1}\left(\sum_{n=k}^{\infty} u_n^q\right)^{\frac{1}{q}}\left(\sum_{j=1}^{k} a_{k,j}^{p'} v_j^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $A \le C \le \bar{C}A$, *where*

$\bar{C} = \left(2(h+1)^q + (h+1)^{2q}\left(1 + h\tilde{C}^q\right)\right)^{\frac{1}{q}}$, $\tilde{C} = \left(1 + \frac{q}{p'}\right)^{\frac{1}{q}}\left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}}$ *and* C *is the best constant in (1).*

Main part. In this section, we present the main results of the article. Here, we give a new pair of equivalent conditions for the Oinarov kernel inequality (1) to hold, as well as lower and upper estimates for the best constant of the inequality. The upper estimate is given as the solution of a nonlinear algebraic equation, which is clearly solved for several values of the parameters p and q. In general, it is proved that there is a unique solution. The lower estimate is said to be better than previously obtained estimates for this type of inequality.

Let us denote:

$$\tilde{A}_1(s)=\left(\sum_{n=s}^{\infty}a_{n,s}^{q}u_n^{q}\right)^{\frac{1}{q}}\left(\sum_{n=1}^{s-1}v_n^{-p'}\right)^{\frac{1}{p'}};$$

$$\tilde{A}_2(s)=\left(\sum_{n=s}^{\infty}u_n^{q}\right)^{\frac{1}{q}}\left(\sum_{n=1}^{s-1}a_{s,n}^{p'}v_n^{-p'}\right)^{\frac{1}{p'}};$$

$$B_1(s)=\left(\sum_{n=s}^{\infty}v_k^{-p'}\left(\sum_{k=n}^{\infty}a_{k,n}^{q}u_k^{q}\right)^{p'}\right)^{\frac{1}{p'q}}\left(\sum_{n=1}^{s}v_n^{-p'}\right)^{\frac{1}{p'q'}};$$

$$B_2(s)=\left(\sum_{n=s}^{\infty}v_n^{-p'}\left(\sum_{k=n}^{\infty}a_{k,n}u_k^{q}\right)^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=s}^{\infty}u_n^{q}\right)^{-\frac{1}{q'}}.$$

Our main result reads:

Theorem 2.1. *Let* $1<p\le q<\infty$ *and* $\{a_{n,k}\}$ *be Oinarov's kernel. Then the inequality (1) holds if and only if*

$$B_1:=\sup_{s\ge1}B_1(s)<\infty,\quad B_2:=\sup_{s\ge1}B_2(s)<\infty. \tag{3}$$

Moreover, the best constant of the inequality (1) satisfies

$$\max\left\{\sup_{s\ge1}\left[\tilde{A}_1^{p'q}(s)+qB_1^{p'q}(s)\right]^{\frac{1}{p'q}},\sup_{s\ge1}\left[\tilde{A}_2^{p'}(s)+B_2^{p'}(s)\right]^{\frac{1}{p'}}\right\}\le C\le X,$$

where X *a positive solution of the corresponding nonlinear equation*

$$X^q-h^{q-1}qB_2X^{q-1}=h^{q-1}q^{\frac{p'+1}{p'}}(q')^{\frac{q-1}{p'}}B_1^{q},\quad if\quad 1<q\le2,$$

$$X^{q'}-h(qB_2)^{\frac{1}{q-1}}X=hq^{\frac{p'+1}{p'(q-1)}}(q')^{\frac{1}{p'}}B_1^{q'},\quad if\quad q\ge2.$$

Proof. *Necessity and the lower estimate.* (3) the necessity of the conditions was published in the 11th series of "*Scientific reports of Bukhara state university*" journal.

Sufficiency and upper estimate. Let us denote

$$S=\sum_{n=1}^{\infty}u_n^{q}\left(\sum_{k=1}^{n}a_{n,k}f_k\right)^{q}$$

then consequently using Lagrange's mean value theorem, Fubini's theorem and Hölder's inequality we get

$$S=\sum_{n=1}^{\infty}u_n^{q}\left\{\sum_{m=1}^{n}\left(\left(\sum_{k=1}^{m}a_{n,k}f_k\right)^{q}-\left(\sum_{k=1}^{m-1}a_{n,k}f_k\right)^{q}\right)\right\}$$

$$= q\sum_{n=1}^{\infty} u_n^q \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m-1} a_{n,k} f_k + \xi_k a_{n,m} f_m\right)^{q-1}\right)$$

$$\le q\sum_{n=1}^{\infty} u_n^q \left(\sum_{m=1}^{n} a_{n,m} f_m \left(\sum_{k=1}^{m} a_{n,k} f_k\right)^{q-1}\right)$$

$$= q\sum_{m=1}^{\infty} (f_m v_m)\left(v_m^{-1}\sum_{n=m}^{\infty} a_{n,m} u_n^q \left(\sum_{k=1}^{m} a_{n,k} f_k\right)^{q-1}\right)$$

$$\le q\left(\sum_{m=1}^{\infty} f_m^p v_m^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} v_m^{-p'}\left(\sum_{n=m}^{\infty} a_{n,m} u_n^q \left(\sum_{k=1}^{m} a_{n,k} f_k\right)^{q-1}\right)^{p'}\right)^{\frac{1}{p'}} = q\|f\|_{l_{p,v}} \bar{S}^{\frac{1}{p'}},$$

where $\xi_k \in (0,1), k = 1, \dots, n,$. So we get the following estimate

$$S \le q\|f\|_{l_{p,v}} \bar{S}^{\frac{1}{p'}}, \qquad (4)$$

where

$$\bar{S} = \sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m\right)^{q-1}\right)^{p'}. \qquad (5)$$

Further, we estimate $\bar{S}$. We divide the proof into two cases.

Case 1. Let $q \ge 2$. Using Oinarov's condition, Minkowski's inequality we obtain the following estimates for the inner sum of $\bar{S}$

$$\left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m\right)^{q-1}\right)^{\frac{1}{q-1}}$$

$$\le h\left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(a_{n,k}\sum_{m=1}^{k} f_m + \sum_{m=1}^{k} a_{k,m} f_m\right)^{q-1}\right)^{\frac{1}{q-1}}$$

[using Minkowski's inequality, $(q \ge 2)$]

$$\le h\left[\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \left(\sum_{m=1}^{k} f_m\right)^{q-1}\right)^{\frac{1}{q-1}} + \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{k,m} f_m\right)^{q-1}\right)^{\frac{1}{q-1}}\right]$$

$$= h\left[\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} f_m\right) + \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} a_{k,m} f_m\right)\right].$$

Then from (5) we get

$$\bar{S} = \sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m\right)^{q-1}\right)^{p'}$$

$$\leq h^{(q-1)p'}\sum_{k=1}^{\infty} v_k^{-p'}\left(\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} f_m\right)+\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} a_{k,m}f_m\right)\right)^{(q-1)p'}$$

$$= h^{(q-1)p'}\left\{\left(\sum_{k=1}^{\infty} v_k^{-p'}\left(\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} f_m\right)\right.\right.\right.$$

$$\left.\left.\left.+\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{\frac{1}{q-1}}\left(\sum_{m=1}^{k} a_{k,m}f_m\right)\right)^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}}\right\}^{(q-1)p'}$$

[using Minkowski's inequality $\left((q-1)p' \geq 1\right)$ we have]

$$\leq h^{(q-1)p'}\left\{\left(\sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{p'}\left(\sum_{m=1}^{k} f_m\right)^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}}\right.$$

$$\left.+\left(\sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{p'}\left(\sum_{m=1}^{k} a_{k,m}f_m\right)^{(q-1)p'}\right)^{\frac{1}{(q-1)p'}}\right\}^{(q-1)p'}$$

$$= h^{(q-1)p'}\left\{S_1^{\frac{1}{(q-1)p'}}+S_2^{\frac{1}{(q-1)p'}}\right\}^{(q-1)p'},$$

i.e.,

$$\bar{S} \leq h^{(q-1)p'}\left\{S_1^{\frac{1}{(q-1)p'}}+S_2^{\frac{1}{(q-1)p'}}\right\}^{(q-1)p'}, \qquad (6)$$

where

$$S_1 = \sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{p'}\left(\sum_{m=1}^{k} f_m\right)^{(q-1)p'}$$

$$S_2 = \sum_{k=1}^{\infty} v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}u_n^q\right)^{p'}\left(\sum_{m=1}^{k} a_{k,m}f_m\right)^{(q-1)p'}.$$

from (4) and (6) we get

$$S \leq q\|f\|_{l_{p,v}}\bar{S}^{\frac{1}{p'}} \leq q\|f\|_{l_{p,v}}h^{q-1}\left(S_1^{\frac{1}{(q-1)p'}}+S_2^{\frac{1}{(q-1)p'}}\right)^{q-1}.$$

Further, we estimate S_1 and S_2 separately.

To estimate S_1, we use discrete Hardy inequality (2) with the exponents $\bar{p} := p, \bar{q} := (q-1)p'$ and weight sequences $\bar{u}_k := v_k^{-p'}\left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q\right)^{p'}$, $\bar{v}_k := v_k$ since

$$S_1^{\frac{1}{(q-1)p'}} = \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \right)^{p'} \left(\sum_{m=1}^{k} f_m \right)^{(q-1)p'} \right)^{\frac{1}{(q-1)p'}} \leq C_{p,(q-1)p'} \left(\sum_{k=1}^{\infty} f_k^p v_k^p \right)^{\frac{1}{p}}. \quad (7)$$

The inequality (7) holds according to Theorem 7 in Chapter 6 of [1] and following estimate

$$B_1^{q'} = \sup_{m \geq 1} \left(\sum_{k=m}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \right)^{p'} \right)^{\frac{1}{(q-1)p'}} \left(\sum_{k=1}^{m} v_k^{-p'} \right)^{\frac{1}{p'}} < \infty$$

we get the estimate for the best constant $C_{p,(q-1)p'}$ in (7)

$$C_{p,(q-1)p'} \leq q^{\frac{1}{p'(q-1)}} (q')^{\frac{1}{p'}} B_1^{q'}.$$

Therefore,

$$S_1^{\frac{1}{(q-1)p'}} \leq q^{\frac{1}{p'(q-1)}} (q')^{\frac{1}{p'}} B_1^{q'} \|f\|_{l_{p,v}}. \quad (8)$$

Now we estimate S_2. Using consequently Fubini's theorem and Minkowskii's inequality we obtain

$$S_2 = \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \right)^{p'} \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{(q-1)p'}$$

$$= \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \right)^{p'} \sum_{l=1}^{k} \left(\left(\sum_{m=1}^{l} a_{k,m} f_m \right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{k,m} f_m \right)^{(q-1)p'} \right)$$

$$= \sum_{l=1}^{\infty} \left(\left(\sum_{m=1}^{l} a_{k,m} f_m \right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{k,m} f_m \right)^{(q-1)p'} \right) \left(\sum_{k=l}^{\infty} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \right)^{p'} v_k^{-p'} \right)$$

$$\leq \sum_{l=1}^{\infty} \left(\left(\sum_{m=1}^{l} a_{k,m} f_m \right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{k,m} f_m \right)^{(q-1)p'} \right) \left(\sum_{k=l}^{\infty} u_k^q \right)^{\frac{p'}{q'}} B_2^{p'}$$

$$= B_2^{p'} \left\{ \left(\sum_{l=1}^{\infty} \left(\left(\sum_{m=1}^{l} a_{k,m} f_m \right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{k,m} f_m \right)^{(q-1)p'} \right) \left(\sum_{k=l}^{\infty} u_k^q \right)^{\frac{p'}{q'}} \right)^{\frac{q'}{p'}} \right\}^{\frac{p'}{q'}}$$

$$= B_2^{p'} \left\{ \sum_{k=1}^{\infty} u_k^q \left(\sum_{l=1}^{k} \left(\left(\sum_{m=1}^{l} a_{k,m} f_m \right)^{(q-1)p'} - \left(\sum_{m=1}^{l-1} a_{k,m} f_m \right)^{(q-1)p'} \right) \right)^{\frac{q'}{p'}} \right\}^{\frac{p'}{q'}}$$

$$= B_2^{p'} \left(\sum_{k=1}^{\infty} u_k^q \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^q \right)^{\frac{p'}{q'}} = B_2^{p'} S^{\frac{p'}{q'}}.$$

From this we get

$$S_2^{\frac{1}{p'(q-1)}} \le B_2^{\frac{1}{q-1}} S^{\frac{1}{q}}. \qquad (9)$$

Taking into account the above estimates, we get

$$S^{\frac{1}{q-1}} \le hq^{\frac{p'+1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} \|f\|_{l_{p,v}}^{q'} + h\left(qB_2\|f\|_{l_{p,v}}\right)^{\frac{1}{q-1}} S^{\frac{1}{q}}. \qquad (10)$$

Now applying Young's inequality to the second term on the right side of (10) and we get

$$S^{\frac{1}{q-1}} \le hq^{\frac{p'+1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} \|f\|_{l_{p,v}}^{q'} + h^q q^{q'-1} B_2^{q'} \|f\|_{l_{p,v}}^{q'} + \frac{S^{\frac{1}{q-1}}}{q'}$$

i.e.,

$$S^{\frac{1}{q}} \le q\left(hq^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p'}} + h^q\right)^{\frac{1}{q'}} B\|f\|_{l_{p,v}},$$

where $B = \max\{B_1, B_2\}$. This means that the conditions $B_1 < \infty$ and $B_2 < \infty$ are sufficient for the inequality (1) to hold. We also get the upper estimate for the best constant C. Using $S^{\frac{1}{q}} \le C\|f\|_{l_{p,v}}$ we rewrite (10) in the form

$$S^{\frac{1}{q-1}} \le hq^{\frac{1}{q-1}} \|f\|_{l_{p,v}}^{\frac{q}{q-1}} \left(q^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + B_2^{\frac{1}{q-1}} C\right),$$

i.e.,

$$\left(\frac{S^{\frac{1}{q}}}{\|f\|_{l_{p,v}}}\right)^{q'} \le hq^{\frac{1}{q-1}} \left(q^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + B_2^{\frac{1}{q-1}} C\right).$$

The following inequality holds for the best constant

$$C^{q'} \le hq^{\frac{1}{q-1}} \left(q^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + B_2^{\frac{1}{q-1}} C\right)$$

From this we obtain

$$\frac{C^{q'}}{q^{\frac{1}{q-1}} \left(q^{\frac{1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + B_2^{\frac{1}{q-1}} C\right)} \le h$$

let us consider the function

$$f(x) = \frac{x^{q'}}{q^{\frac{p'+1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + (qB_2)^{\frac{1}{q-1}} x}$$

corresponding to the left-hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in $(0, \infty)$, $f(0) = 0$ and $f(\infty) = \infty$. Which implies that the equation $f(x) = h$ has exactly one positive solution in $(0, \infty)$. If X is a solution of the equation, which means that

$$X^{q'} = h\left(q^{\frac{p'+1}{p'(q-1)}}(q')^{\frac{1}{p'}} B_1^{q'} + (qB_2)^{\frac{1}{q-1}} X\right),$$

tthen $C \le X$.

Case 2. Let $1 < q \le 2$ Using the Oinarov's condition, we have that

$$\bar{S}_1 = \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m \right)^{q-1} \le h^{q-1} \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(a_{n,k} \sum_{m=1}^{k} f_m + \sum_{m=1}^{k} a_{k,m} f_m \right)^{q-1}$$

$$\le h^{q-1} \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\left(a_{n,k} \sum_{m=1}^{k} f_m \right)^{q-1} + \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{q-1} \right)$$

$$= h^{q-1} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \left(\sum_{m=1}^{k} f_m \right)^{q-1} + \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{q-1} \right)$$

Putting this expression in (5), we apply the Minkowski's inequality and obtain the following estimate

$$\bar{S} = \sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{n,m} f_m \right)^{q-1} \right)^{p'}$$

$$= h^{(q-1)p'} \left\{ \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \left(\sum_{m=1}^{k} f_m \right)^{q-1} + \sum_{n=k}^{\infty} a_{n,k} u_n^q \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{q-1} \right)^{p'} \right)^{\frac{1}{p'}} \right\}^{p'}$$

$$\le h^{(q-1)p'} \left\{ \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k}^q u_n^q \right)^{p'} \left(\sum_{m=1}^{k} f_m \right)^{(q-1)p'} \right)^{\frac{1}{p'}} \right.$$

$$\left. + \left(\sum_{k=1}^{\infty} v_k^{-p'} \left(\sum_{n=k}^{\infty} a_{n,k} u_n^q \right)^{p'} \left(\sum_{m=1}^{k} a_{k,m} f_m \right)^{(q-1)p'} \right)^{\frac{1}{p'}} \right\}^{p'}$$

$$= h^{(q-1)p'} \left\{ S_1^{\frac{1}{p'}} + S_2^{\frac{1}{p'}} \right\}^{p'}$$

i.e.,

$$\bar{S} \le h^{(q-1)p'} \left\{ S_1^{\frac{1}{p'}} + S_2^{\frac{1}{p'}} \right\}^{p'}$$

From (8) and (9) and we have, respectively, the following

$$S_1^{\frac{1}{p'}} \le q^{\frac{1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^{q-1}$$

and

$$S_2^{\frac{1}{p'}} \le B_2 S^{\frac{1}{q'}}.$$

According to (4) we get the following estimate:

$$S \le q \|f\|_{l_{p,v}} h^{q-1} \left(q^{\frac{1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^{q-1} + B_2 S^{\frac{1}{q'}} \right) \qquad (11)$$

i.e.,

$$S \le h^{q-1} q^{\frac{1}{p'}+1} (q')^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^q + h^{q-1} q B_2 \|f\|_{l_{p,v}} S^{\frac{1}{q'}}.$$

Applying Young's inequality to the last term of the above sum, we get the following inequality

$$S \le h^{q-1} q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^q + \frac{\left(h^{q-1} q B_2 \|f\|_{l_{p,v}}\right)^q}{q} + \frac{S}{q'}$$

i.e.,

$$S \le \left(h^{q-1} q^{\frac{2p'+1}{p'}} (q')^{\frac{q-1}{p'}} + (h^{q-1} q)^q \right) B^q \|f\|_{l_{p,v}}^q,$$

where $B = \max\{B_1, B_2\}$ from this we obtain

$$S^{\frac{1}{q}} \le \left(h^{q-1} q^{\frac{2p'+1}{p'}} (q')^{\frac{q-1}{p'}} + (h^{q-1} q)^q \right)^{\frac{1}{q}} B \|f\|_{l_{p,v}}.$$

This also means that the condition $B_1 < \infty$ and $B_2 < \infty$ are sufficient for the inequality to hold . Using $S^{\frac{1}{q}} \le C\|f\|_{l_{p,v}}$ we rewrite (11) in the following form

$$S \le q \|f\|_{l_{p,v}} h^{q-1} \left(q^{\frac{1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q \|f\|_{l_{p,v}}^{q-1} + B_2 \left(C\|f\|_{l_{p,v}} \right)^{q-1} \right)$$

i.e.,

$$\frac{S}{\|f\|_{l_{p,v}}^q} \le h^{q-1} \left(q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q + q B_2 C^{q-1} \right).$$

Then we have the estimate for the best constant

$$C^q \le h^{q-1} \left(q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q + q B_2 C^{q-1} \right).$$

Consequently, we obtain

$$\frac{C^q}{\left(q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q + q B_2 C^{q-1} \right)} \le h^{q-1}$$

let us consider the function

$$f(x) = \frac{x^q}{\left(q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q + q B_2 x^{q-1} \right)}$$

It is to see that this function is monotone increasing and continuous in $(0, \infty)$ and the equation $f(x) = h^{q-1}$ has one positive solution in $(0, \infty)$. If X is a solution of the equation, which means

$$X^q = h^{q-1} \left(q^{\frac{p'+1}{p'}} (q')^{\frac{q-1}{p'}} B_1^q + q B_2 X^{q-1} \right)$$

then $C \le X$. The proof is complete.

Conclusion. In the theory of Hardy's inequalities, there are many equivalent conditions that ensure that the inequality holds. The main goal is to obtain more accurate estimates for the best inequality constant of the inequality. This is particularly well studied in cases where the kernel of the inequality is a constant. Therefore, more accurate estimates for the constants of such inequalities are common.

But if the kernel is different from the constant, then the problem becomes more complicated. Even estimates for the best constant of Hardy-type inequalities with kernel were almost non-existent until recent years. Of course, not taking into account that only one-sided estimates were obtained. In the last five years, works on this topic have started to appear. The initial results were organized for integral inequalities with Oinarov's kernel.

Also in this work, new equivalent conditions for the discrete Hardy inequality with Oinarov's kernel are given. The obtained results can be used in spectral analysis and oscillation theory.

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