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PERIHELION SHIFT AND LIGHT BENDING BY A GRAVITATIONAL OBJECT IN GENERAL RELATIVITY

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Abstract. In the present research note, we have derived the explicit expression for the perihelion precession and the deflection angle of light by the gravitational object. The explicit expression for the trajectory of the planet orbiting around a central object in the framework of Newtonian theory as well as general relativity has been discussed. It is studied the deflection of the light ray and gravitational lensing effects by the gravitational object in the weak field approximation. Using the gravitational lensing equation the magnification of the primary and secondary images has been derived. It is shown that for small angle β, it is difficult to distinguish the magnifications of the primary and secondary images, while for the largest value of the angle β the magnification of the primary image dominates will equal to the total magnification. The time delay of the light ray passing through the gravitational object is also discussed. Finally, the classical expression for the energy of relativistic particles has been derived in the frame of general relativity.

Keywords: general relativity, Perihelion Precession, deflection of light, gravitational lens, Einstein's Theory, Mercury's orbit, gravitational field, geodesics, spacetime curvature, Schwarzschild Metric, gravitational redshift, light bending, relativistic rffects, astronomical observations, Einstein Field Equations, Keplerian Motion, orbital mechanics, Solar System tests, Black Hole gravity, cosmological implications.

Umumiy nisbiylik nazariyasida tortishish maydonida perigeliynning siljishi va yorug‘lik nurlanishining burilishi

Annotatsiya. Ushbu tadqiqot ishida biz gravitatsion obyekt tomonidan perihelionning siljishi va yorug‘likning og‘ish burchagi uchun aniq ifodani keltirib chiqdik. Markaziy obyekt atrofida sayyoraning trayektoriyasi Nyuton nazariyasi va umumiy nisbiylik doirasida muhokama qilindi. Yorug‘lik nuri og‘ishi va gravitatsion linzalash effektlari kuchsiz maydon taxminlari asosida o‘rganildi. Gravitatsion linzalash tenglamasidan foydalanib, birlamchi va ikkilamchi tasvirlarning kattalashish darajasi chiqarildi. Ko‘rsatildi-ki, kichik burchak β uchun birlamchi va ikkilamchi tasvirlarning kattalashishini ajratish qiyin, katta β burchak qiymatlarida esa birlamchi tasvirning kattalashishi ustun kelib, umumiy kattalashish bilan teng bo‘ladi. Yorug‘lik nurlarining gravitatsion obyekt orqali o‘tish vaqt kechikishi ham muhokama qilindi. Nihoyat, umumiy nisbiylik doirasida relativistik zarrachalarning energiyasi uchun klassik ifoda keltirildi.

Kalit so‘zlar: umumiy nisbiylik, perihelionning siljishi, yorug‘likning og‘ishi, gravitatsion linza, Eynshteyn nazariyasi, Merkuriy orbiti, gravitatsion maydon, geodezikalar, fazo-vaqt

egilishi, Shvartsshild, gravitatsion qizil siljish, yorug‘likning burilishi, relativistik effektlar, astronomik kuzatuvlar, Eynshteyn tenglamalari, Kepler harakati, orbital mexanika, Quyosh tizimi sinovlari, qora tuynuk gravitatsiyasi, kosmologik natijalar.

Смещение перигелия и искривление света гравитационным объектом в общей теории относительности

Аннотация. В данной исследовательской заметке мы вывели явное выражение для прецессии перигелия и угла отклонения света гравитационным объектом. Обсуждена явная траектория планеты, обращающейся вокруг центрального объекта, в рамках ньютоновской теории и общей теории относительности. Исследованы отклонение светового луча и эффекты гравитационного линзирования в приближении слабого гравитационного поля. С помощью уравнения гравитационного линзирования были получены коэффициенты увеличения первичного и вторичного изображений. Показано, что при малом угле β трудно различить увеличения первичного и вторичного изображений, тогда как при большем значении угла β увеличение первичного изображения доминирует и становится равным общему увеличению. Также обсуждается временная задержка светового луча при его прохождении через гравитационный объект. Наконец, в рамках общей теории относительности выведено классическое выражение для энергии релятивистских частиц.

Ключевые слова: Общая теория относительности, прецессия перигелия, отклонение света, гравитационная линза, теория Эйнштейна, орбита Меркурия, гравитационное поле, геодезические линии, кривизна пространства-времени, метрика Шварцшильда, гравитационное красное смещение, искривление света, релятивистские эффекты, астрономические наблюдения, уравнения Эйнштейна, кеплеровское движение, орбитальная механика, тесты солнечной системы, гравитация чёрных дыр, космологические последствия.

I. Introduction. Based on astronomical observations, in the early 17th century, Kepler determined that the trajectory traced by a planet in the solar system is an ellipse, with the Sun situated at one of its focal points. Indeed, Kepler's laws are derived within the framework of Newtonian mechanics. However, this theory fails to account for the orbital behavior of certain planets in the solar system, particularly Mercury's motion. This limitation underscores the success of general relativity, a theory of gravitation proposed by Einstein in 1915, which aligns remarkably well with numerous astronomical observations and experiments conducted on a solar system scale.

Specifically, three pivotal tests of general relativity-the precession of Mercury's perihelion [1, 2], the bending of light by the Sun [3, 4], and the radar time delay experiment [5, 6]-have all substantiated Einstein's gravitational theory within the bounds of observational and experimental accuracy. As previously noted, gravitational lensing stands out as one of the most significant validations of general relativity. Although this phenomenon has been observed in distant astrophysical sources, these observations are often imprecise and provide limited constraints on general relativity.

The most accurate tests are akin to Eddington's 1919 experiment, measuring the deflection of radiation from a distant source caused by the Sun. Distant radio sources, particularly quasars known for their intense radio emissions, allow for precise analysis. A significant advancement in positional accuracy has been achieved through the integration of radio telescopes across the globe, a method known as very long baseline interferometry (VLBI). This technique utilizes the phase information of

radio signals detected by telescopes separated by vast distances. Recently, these radio telescopes have measured the deflection of radio waves by the Sun with exceptional precision, confirming the deflection predicted by general relativity to within 0.03% [7].

Launched in 2013, the Gaia spacecraft carries out a detailed survey of one billion stars within the Milky Way, determining their positions with an accuracy of 24 microarcseconds. This also enables stringent new tests of the Sun's gravitational deflection of light, as anticipated by general relativity.

In this paper, we derive the formula for the perihelion shift of a planet within the frameworks of both Newtonian mechanics and general relativity. The structure of the paper is as follows: In Sect. II, we present a comprehensive derivation of the perihelion shift of a planet orbiting a gravitational mass under both Newtonian mechanics and general relativity. Sect. III explores the deflection of light rays and the gravitational lensing phenomenon within the scope of general relativity. Finally, Sect. IV summarizes the findings.

II. Perihelion precession. The perihelion represents the point in a planet's orbit where it is closest to the central celestial body. Given its significance in numerous applications, analyzing the trajectories of massive or massless particles in various geometries derived as solutions to Einstein's gravitational field equations and their generalizations is a critical area in the study of general relativity. The Schwarzschild solution, the first exact resolution to the vacuum field equations, has proven highly effective in accurately describing all astronomical phenomena within the Solar System's scale.

The precise equation of motion in Schwarzschild geometry is notably nonlinear, making it necessary to employ approximate methods to determine observable physical quantities. Even the first-order approximation of the equation of motion provides an accurate estimation of Mercury's perihelion precession and the Sun's gravitational bending of light. Nevertheless, owing to the complexity and relevance of this problem, advanced mathematical techniques can be applied to extend the analysis to higher-order approximations, enhancing the understanding of planetary motions and the deflection of light.

A. Newtonian approach. In Newtonian theory, the Lagrangian for a celestial object of mass m influenced by the central gravitational object of mass M at the equatorial plane is given by [8]

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GMm}{r}, \tag{1}$$

where an overdot denoted the derivative with respect to time, G is the Newtonian gravitational constant. The constants of motion, namely, energy, E, and and angular momentum, L, of the celestial object can be easily found as

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r}, L = mr^2\dot{\phi}, \tag{2}$$

Using the above equations (2) and (2) one can obtain

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} = \frac{2E}{mL^2} + \frac{2GM}{L^2 r} \tag{3}$$

Keep in mind that the solution to the above equation represents the trajectory of the celestial object around central object. Hereafter intruding the new coordinate $u = 1/r$ and differentiating the equation (3), one can have [8]

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{L^2}, \tag{4}$$

where u is a function of the angle ϕ, i.e $u = u(\phi)$. Note that equation (4) reduces to the standard harmonic oscillator equation for $(u - GMm^2/L^2)$ and the solution is given by $u - GMm^2/L^2 = A\cos(\phi - \phi_0)$, where A and ϕ_0 are the constants of integration, respectively. For

simplicity, we set $\phi_0 = 0$ and $A = eGMm^2/L^2$. Consequently, the solution to equation (4) is simply can be found as [8]

$$u(\phi) = \frac{GMm^2}{L^2}(1 + e\cos\phi) \tag{5}$$

where e is the eccentricity.

The precession of an orbit looks at the motion of the perihelion over time. The perihelion of an orbit is the point closest to the host body. In Figure 1 shows the perihelion of the point particle orbiting around a central gravitational object. As one can see that in absence of eccentricity trajectory of point particle follows circular orbits, while for the largest values of the eccentricity trajectory will be elliptical.

B. General relativistic approach. Now we focus on the same problem in the framework of the general relativity. To take general relativistic effect, we consider the Schwarzschild spacetime which is given by the following line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) d(ct)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

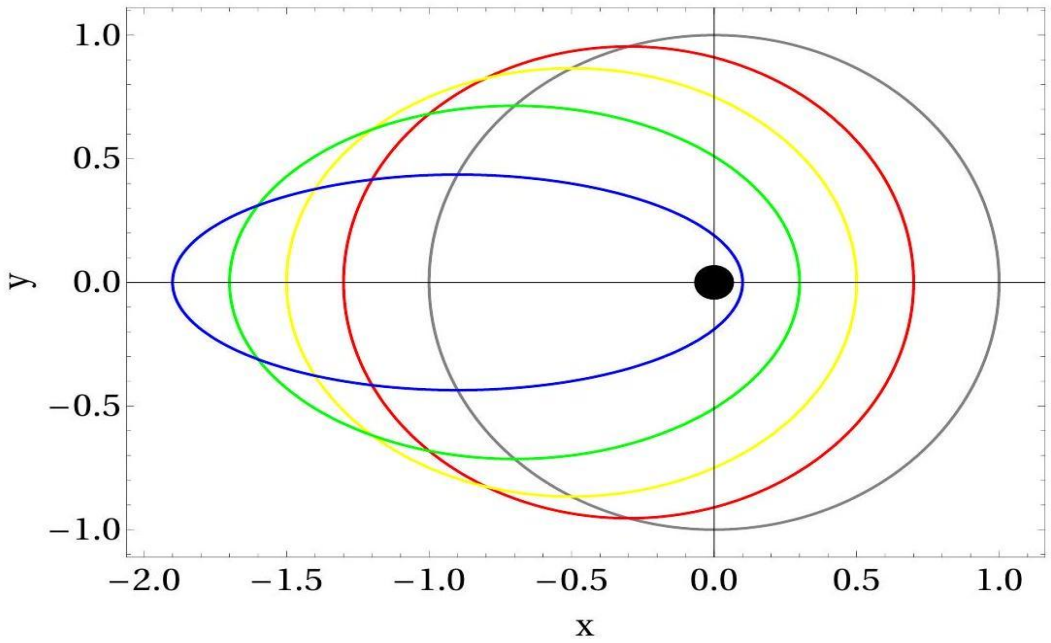

FIG. 1: The trajectories of particle for different values of eccentricity, in particular, $\mathbf{e = 0, 0.3, 0.5, 0.7, 0.9}$.

where c is the speed of light, M is the total mass of the black hole and the virtual size of the black hole is represented by the Schwarzschild radius $r_g = 2GM/c^2$.

In fact, the motion of test particle around the black hole is described by the geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{7}$$

where $\dot{x}^\alpha = dx^\alpha/d\tau$ is the 4-velocity of test particle normalized as $\dot{x}_\alpha\dot{x}^\alpha = -1, \Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols. Hereafter integrating equation (7) at the equatorial plane, one can obtain

$$\frac{dt}{d\tau} = \frac{E}{mc^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \frac{d\phi}{d\tau} = \frac{L}{mr^2}, \left(\frac{dr}{d(c\tau)}\right)^2 = \left(\frac{E}{mc^2}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)\left[1 + \left(\frac{L}{mcr}\right)^2\right] \tag{8}$$

Taking into account equations in (8), and after introducing new variable $u = 1/r$, the radial equation read

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{mc}{L}\right)^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] + \frac{2GMm^2}{L^2}u - u^2 + \frac{2GM}{c^2}u^3 \tag{9}$$

Hereafter differentiation equation (9), one can get

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}u^2. \tag{10}$$

Here the last term in equation (10) is responsible for general relativistic correction to the equation for the trajectory of the test particle. On the other hand, from the mathematical point of view, equation (10) represents the nonlinear harmonic oscillator equation and the solution is defined by the incomplete elliptic integral (See, e.g. []). This solution is not allowed to find perihelion shift. That is why the semi-analytical approach can be performed to find approximate value of the perihelion shift. Let's introduce the expansion parameter $\epsilon = 3(GMm/cL)^2 \ll 1$ which allows to expand the solution in the form:

$$u(\phi) = u_0(\phi) + \epsilon u_1(\phi) + \mathcal{O}(\epsilon^2). \tag{11}$$

Inserting the equation (11) into (10), one can obtain solution in the zeroth-order approximation in the form:

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GMm^2}{L^2}, \rightarrow\ u_0(\phi) = \frac{GMm^2}{L^2}(1 + e\cos\phi) \tag{12}$$

which is the same as predicted in the Newtonian theory, while in the first-order approximation equation (10) reads

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{L^2}{GM}u_0^2 = \frac{GMm^2}{L^2}\left(1 + \frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi\right). \tag{13}$$

The general solution to equation (13) can be found as $u_1(\phi) = A + B\phi\sin\phi + C\cos 2\phi$, which satisfies the following equation

$$\frac{d^2u_1}{d\phi^2} + u_1 = A + 2B\cos\phi - 3C\cos 2\phi \tag{14}$$

where A, B, and C are unknown constants found by comparing equations (13) and (14) in the form:

$$A = \frac{GMm^2}{L^2}\left(1 + \frac{e^2}{2}\right), B = \frac{GMm^2e}{L^2}, C = -\frac{GMm^2e^2}{6L^2}. \tag{15}$$

Before presenting the final result for u, one can use the following useful expression for small ϵ in the form:

$$\cos[\phi(1-\epsilon)] = \cos\phi\cos\epsilon\phi + \sin\phi\sin\epsilon\phi \simeq \cos\phi + \epsilon\phi\sin\phi + \mathcal{O}(\epsilon^2) \tag{16}$$

Consequently, taking into account all facts above together with equation (16), the solution (11) can rewritten as

$$u(\phi) = \frac{GMm^2}{L^2}[1 + e\cos\phi(1-\epsilon)] + \frac{\epsilon GMm^2}{L^2}\left[1 + \frac{e^2}{3}(1 + \sin^2\phi)\right]. \quad (17)$$

As the precession is the result of the orbit not being periodic in 2π, the precession must result from this term, and so can be calculated from it. Take the precession as $\delta\phi$. This then gives $u(0) = u(2\pi + \delta\phi)$. The perihelia first occurs at $\phi = 0$, as was defined earlier. This means that the second perihelia will occur when the cos term generating the precession has gone through a full 2π. Then, one can obtain the following relation $2\pi = (2\pi + \delta\phi)(1-\epsilon)$ which means that

$$\delta\phi = \frac{2\pi\epsilon}{1-\epsilon} \simeq 2\pi\epsilon = \frac{6\pi G^2M^2m^2}{c^2L^2}. \quad (18)$$

Figure 2 illustrates the perihelion precession of point particle in general relativity for the different values of the eccentricity.

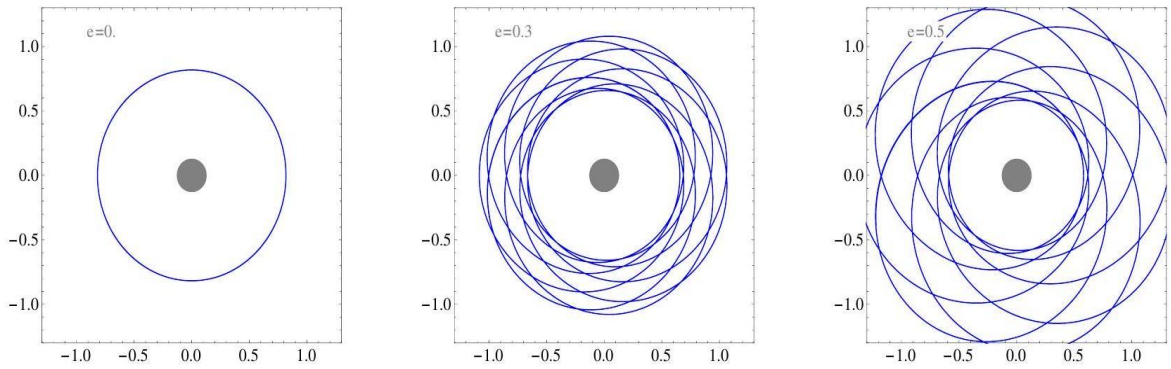

FIG. 2: The perihelion shift of point particle for different values of eccentricity.

III. Deflection of light. In the weak gravitational field approximation the metric tensor of the spacetime can be written as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, where $\eta_{\mu\nu}$ is the metric tensor in flat space, $h_{\mu\nu}$ is a small perturbation $\eta_{\alpha\beta} = \eta^{\alpha\beta}, h_{\alpha\beta} = h^{\alpha\beta}, h_{\alpha\beta}h^{\alpha\beta} \to 0$. According to Ref. [9], the deflection angle of photon is defined as the difference between the directions of the incoming and of the outgoing light rays. We can write the expression of the deflection angle of the light ray: $\hat{\alpha} = \mathbf{e}_{\text{out}} - \mathbf{e}_{\text{in}}$, where $\mathbf{e}_{\text{in}}$ and $\mathbf{e}_{\text{out}}$ are the unit vectors along the spatial component momentum vector $\mathbf{p}$ of the "incoming" and "outgoing" photon, respectively, i.e. $\mathbf{e} = \mathbf{p}/p$, and $p = \sqrt{p_x^2 + p_y^2 + p_z^2} = p_z$. The explicit expression for the deflection angle is given by [9]

$$\hat{\alpha}_b = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{d}{db}(h_{tt} + h_{zz})dz \quad (19)$$

with h_{tt} and h_{zz} in the Schwarzschild spacetime are [8]

$$h_{tt} = \frac{2GM}{c^2r}, h_{ij} = \frac{2GM}{c^2r}\hat{n}_i\hat{n}_j, h_{zz} = \frac{2GM}{c^2r}\cos^2\theta \quad (20)$$

where $\hat{n}_i$ is the component of the unit vector with the same direction as the radius vector $r_i = (x, y, z)$ and has the form $\hat{n}_i = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. Using equations (19) and (20), the expression for the deflection angle of a light ray passing near a static compact object can be found as

$$\hat{\alpha}_b = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{d}{db}\left[\frac{2GM}{c^2\sqrt{b^2+z^2}}\left(1+\frac{z^2}{b^2+z^2}\right)\right]dz = \frac{4GM}{c^2 b} \tag{21}$$

where b is the impact parameter of the light ray. While in next leading approximation the deflection of angle can be found as

$$\hat{\alpha}_b = \frac{4GM}{c^2 b} + \pi\left(\frac{GM}{c^2 b}\right)^2 \tag{22}$$

Now, we study the observational consequences of gravitational lensing, namely the magnification of image sources, Einstein cross (rings), time delay, etc. For this purpose, one can e the lens equation which relates the angle β between the real position of the source and lens with respect to the observer, the angle θ between the apparent image of the source and the observer-lens axis, and the deflection angle α: $\beta = \theta - \Theta_0^2/\theta$, where

$\Theta_0 = \sqrt{4GM\,D_{ls}/c^2 D_l D_s}$ is the Einstein ring, with D_s, D_{ls} and D_l being the distances between observer and the source, lens and the source, and observer and lens, respectively. Let us now consider the image magnification due to lensing. The solution of lens equation can be found as $\theta = \left(\beta \pm \sqrt{\beta^2 + 4\Theta_0^2}\right)/2$. The magnification of the image is identified as $\mu = |(\theta/\beta)(d\theta/d\beta)|$, while for individual images one can have

$$\mu_1 = \frac{1}{4}\left[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} + 2\right], \mu_2 = \frac{1}{4}\left[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} - 2\right], y = \beta/\Theta_0 \tag{23}$$

where the sub indecies " 1 " and " 2 " denote the primary and secondary images of the source, respectively. The total $\mu = \mu_1 + \mu_2$ and the ratio $R = \mu_1/\mu_2$ of magnifications of images are given by

$$\mu = \frac{y^2+2}{y\sqrt{y^2+4}}, R = \left(\frac{\sqrt{y^2+4}+y}{\sqrt{y^2+4}-y}\right)^2 \tag{24}$$

Figure 3 depicts the magnification of the primary and secondary images of a source caused by weak gravitational lensing: it illustrates the splitting of the first and second images, often referred to as the Einstein cross, under the influence of an external magnetic field, as compared to the image observed in a plasma medium. For "positive parity," the Einstein cross appears larger, whereas for "negative parity," its size is reduced relative to the plasma scenario.

When a fluctuating source is situated behind the lensing mass, it produces a detectable fluctuating image. However, the variations in the source and its corresponding image do not necessarily happen at the same moment; typically, there is a temporal offset between them due to two main factors. First, there is a purely geometric time lag. Second, there is a delay caused by the gravitational field of the lensing mass, often termed the Shapiro delay. Consequently, the total time delay arises from the combined influence of the spatial configuration and the gravitational potential.

$$\Delta T = \frac{4GM}{c^3}(1+z)\left[\ln\left(\frac{\sqrt{y^2+4}+y}{\sqrt{y^2+4}-y}\right) + \frac{1}{2}y\sqrt{y^2+4}\right]. \tag{25}$$

where z is the redshift of the lensing object. To have an idea about the value of the time delay for the suppermassive black hole of mass $10^6 M_\odot$ can be estimated as

$$\Delta T \sim 2\times 10^{-4}\left(\frac{M}{10^6 M_\odot}\right)\text{s} \tag{26}$$

Dependence of the time delay due to the gravitational field from the position y for the value of the red-shift $z = 0$ and $z = 2.7$ is illustrated in Fig. 4. It is easy to see that with increasing of β angle the time delay light ray increases.

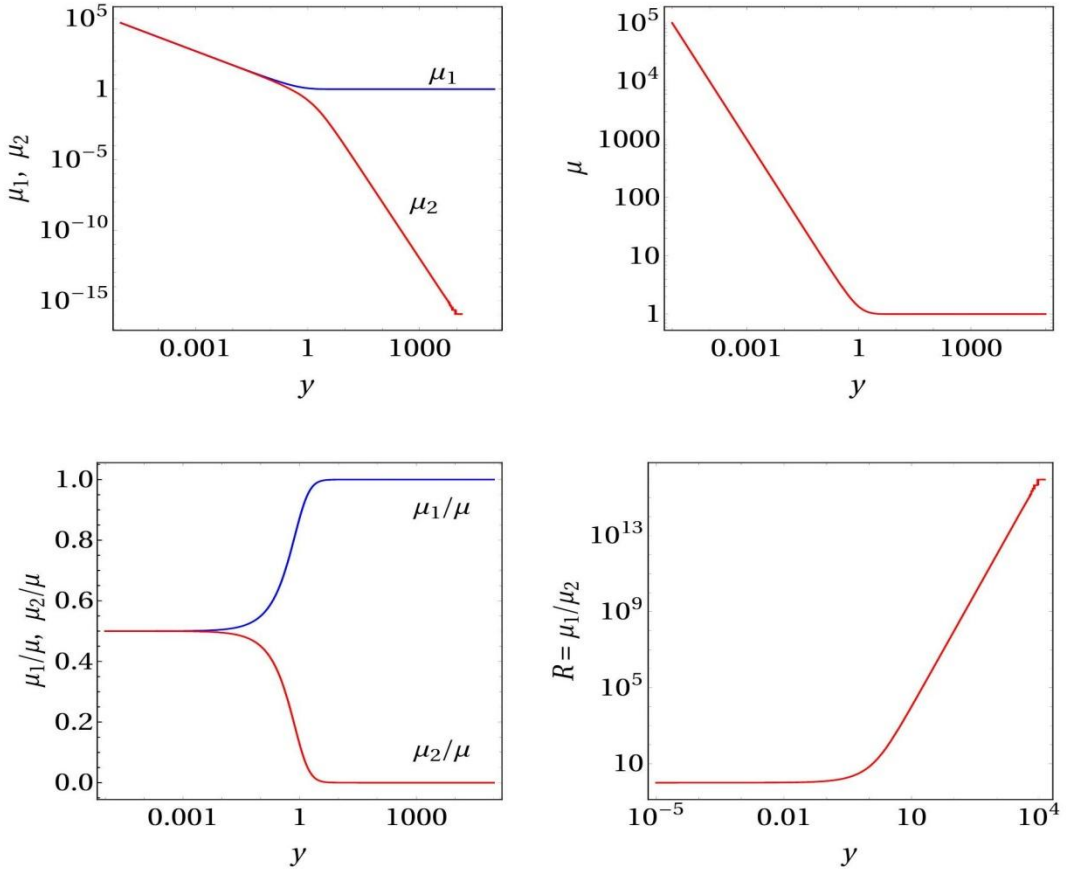

FIG. 3: The magnification of individual image (left) the total magnification (centre) and the ratio of magnifications (right) as the functions of fractinal angle $\boldsymbol{y = \beta/\Theta_0}$.

A. From general relativity to special relativity

The Lagrangian for test particle of mass m is given by

$$\mathcal{L} = \frac{1}{2} m g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta}, \dot{x}^{\alpha} = \frac{dx^{\alpha}}{d\lambda} \tag{27}$$

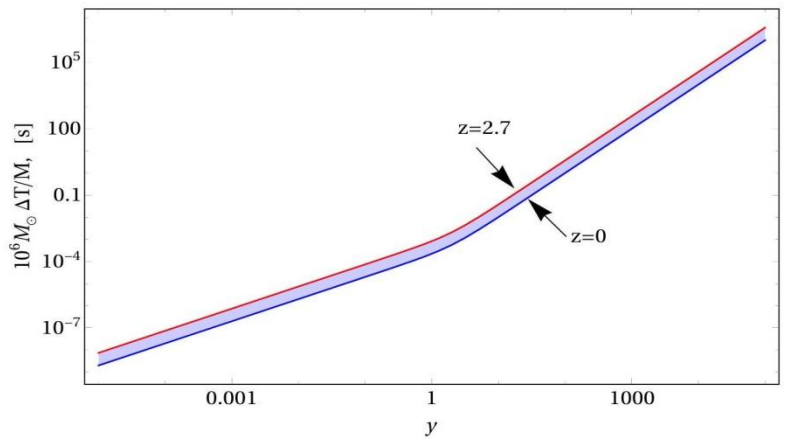

FIG. 4: Time delay ΔT of the light-ray as a function of the angle $y = \beta/\Theta_0$.

where $g_{\alpha\beta}$ is the metric tensor, λ is an affine parameter, $\dot{x}^{\alpha}$ is the four-velocity normalized as, $g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = -1$, that allows to write the following expression:

$$g_{tt}\dot{t}^2(1 - v^2/c^2) = -1 \tag{28}$$

where v is the relative three-velocity of particle measured in a frame of a local observer defined as, $v^2 = v_{\hat{r}}^2 + v_{\hat{\theta}}^2 + v_{\hat{\phi}}^2$, and the orthonormal components are given as [8,10,11]

$$v_{\hat{r}} = \frac{dr}{dt}\sqrt{\frac{g_{rr}}{-g_{tt}}}, v_{\hat{\theta}} = \frac{d\theta}{dt}\sqrt{\frac{g_{\theta\theta}}{-g_{tt}}}, v_{\hat{\phi}} = \frac{d\phi}{dt}\sqrt{\frac{g_{\phi\phi}}{-g_{tt}}} \tag{29}$$

The conserved quantities, namely, the energy and angular momentum of test particle are given by

$$P_t = -E = mc^2 g_{tt}\dot{t}, P_{\phi} = L = m g_{\phi\phi}\dot{\phi}. \tag{30}$$

Hereafter eliminating, $\dot{t}$, from equations (30) and (28), the classical expression for the energy of relativistic particle in the frame of GR can be expressed as [8, 10, 11,12,13]

$$E = \frac{mc^2\sqrt{-g_{tt}}}{\sqrt{1 - v^2/c^2}}, L = \frac{m v_{\phi}\sqrt{g_{\phi\phi}}}{\sqrt{1 - v^2/c^2}}. \tag{31}$$

On the other hand, using the normalization of the four-velocity particle, one can obtain an expression for the effective potential in the form:

$$V_{\text{eff}} \equiv -g_{tt}\left(1 + \frac{(L/mc)^2}{g_{\phi\phi}}\right) = \left(\frac{E}{mc^2}\right)^2 + g_{tt}\left(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2\right) \tag{32}$$

The critical values for the conserved quantities can be found, by applying the conditions $\dot{r} = \ddot{r} = 0$ and $\dot{\theta} = \ddot{\theta} = 0$, as

$$E = \frac{mc^2\sqrt{-g_{tt}}}{\sqrt{1 - \frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}}}, L = \frac{mc\sqrt{g_{\phi\phi}}}{\sqrt{1 - \frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}}\sqrt{\frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}}}, \tag{33}$$

where prime denotes the derivative with respect to the radial coordinate.

Now, comparing the energy expressions (31) and (33), one can express the orbital velocity of a particle in the form:

$$v = v_{\phi} = c\sqrt{\frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}} = c\sqrt{\frac{\partial_r \ln g_{tt}}{\partial_r \ln g_{\phi\phi}}}, \tag{34}$$

Which can be readily determined for any spacetime geometry. It is straightforward to demonstrate that the orbital velocity of a particle at the photon sphere always matches the speed of light, i.e., $v = c$. This follows from the fact that the position of the photon sphere is obtained by setting the denominator of the energy expression (33) to zero.

IV. Conclusions. The explicit expression for the trajectory of the planet orbiting around a central object in the framework of Newtonian theory has been reproduced. The dependence of the shape of the trajectory of the planet from the eccentricity has been explicitly analyzed. It shown that with increasing the eccentricity the perihelion of the planet come close to the central object.

The same issue has been studied within the framework of general relativity. It has been demonstrated that a planet moves around a gravitational body in an elliptical orbit; however, the perihelion of the planet continuously shifts with each revolution, a phenomenon first predicted by the general theory of relativity. Additionally, it has been shown that the planet's trajectory can be described using elliptic integrals, but these do not provide insight into the perihelion shift of the planet's orbit around the gravitational object. It is studied the deflection of the light ray and gravitational lensing effects by the gravitational object in the weak field approximation. To involve general relativistic effects the Schwarzshild spacetime has been considered. The derivation of the expression for the deflection angle has been explicitly shown in the first and second-order approximations. Using the gravitational lensing equation the magnification of the primary and secondary images has been derived. It is shown that for small angle β, it is difficult to distinguish the magnifications of the primary and secondary images, while for the largest value of the angle β the magnification of the primary image dominates will equal to the total magnification. The time delay of the light ray passing through the gravitational object is also discussed.

Finally, it investigates particle motion around the gravitational compact object described by arbitrary spacetime. The analog of special relativity has been shown by considering the pure general relativistic approach.

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